Expectation of sums of Random Variables

$Z = g(X,Y)$ What $E[Z] = E[g(X,Y)]$

Lemma: If X and Y are discrete with joint mass function $p(x,y)$

Then $E[g(X,Y)] = \sum_y \sum_x g(x,y)\, p(x,y)$

If X and Y are continuous with joint density function $f(x,y)$

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f(x,y)\,dx\,dy$$

Proof is similar to the single veriable case

In case $g \geq 0$ is nonnegative

$$E[g(X,Y)] = \int_0^{\infty} P\{g(X,Y) > t\}\,dt$$

$$= \int_0^{\infty} \iint_{g(x,y)>0} f(x,y)\,dx\,dy\,dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{g(x,y)} f(x,y)\,dt\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f(x,y)\,dx\,dy$$

Corollary: $E[X+Y] = E[X] + E[Y]$

Already proved in discrete case

In continuous case, we have joint density function $f(x,y)$

$$E[X+Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y) f(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\,dx\,dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f(x,y)\,dx\,dy$$

Recall: $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

$$= \int_{-\infty}^{\infty} x\left[\int_{-\infty}^{\infty} f(x,y)\,dy\right]dx + \int_{-\infty}^{\infty} y\left[\int_{-\infty}^{\infty} f(x,y)\,dx\right]dy$$

$$= \int_{-\infty}^{\infty} x f_X(x)\,dx + \int_{-\infty}^{\infty} y f_Y(y)\,dy$$

$$= E[X] + E[Y]$$

Does not require independence.

Corollary if $X \geq Y$ (always)

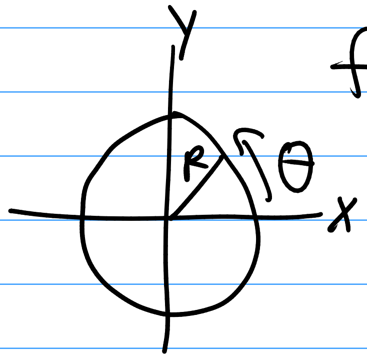
then $E[X] \geq E[Y]$

Proof $X - Y \geq 0$ always

$$E[X-Y] \geq 0 \Rightarrow E[X] - E[Y] \geq 0$$

e.g. $\int (x-y) f(x,y)\, dx\, dy \geq 0$ or $\sum_{x,y} (x-y) p(x,y) \geq 0$

Example : uniform distribution of a disk of radius R



$$f(x,y) = \begin{cases} \frac{1}{\pi R^2} & \text{if } x^2+y^2 \leq R^2 \\ 0 & \text{otherwise} \end{cases}$$

What is the expected distance from the point (X,Y) to the center $(0,0)$?

$$D = \sqrt{X^2+Y^2}$$

$$E[D] = \iint \sqrt{x^2+y^2}\, f(x,y)\, dx\, dy$$

Polar coordinates r, θ $\quad r = \sqrt{x^2+y^2}$

$$dx\,dy = r\,dr\,d\theta$$

$$E[D] = \iint r\, f(x,y)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^R r^2 \frac{1}{\pi R^2}\, dr\, d\theta = \frac{1}{\pi R^2}(2\pi)\frac{R^3}{3}$$

$$= \frac{2}{3}R$$

Sometimes its useful to undrstand a given R.V. as a sum of simpler ones

X binomial w/ parameters n, p ← prob. of success

↗ # trials

let $X_i = \begin{cases} 1 & \text{if ith trial is a success} \\ 0 & \text{otherwise} \end{cases}$

Then $X = X_1 + X_2 + \cdots + X_n$

Note X_i is a "0 or 1" Random Variable

$$E[X_i] = 0\, P\{X_i = 0\} + 1\, P\{X_i = 1\} = P\{X_i = 1\}$$

For "0 or 1" RV $E[X] = P\{X=1\}$

$$E[X] = E[X_1] + \cdots + E[X_n]$$

$$= p + p + \cdots + p = np$$

Recall Hypergeometric distribution

N balls in urn m are white $N-m$ are black

Select n balls without replacement

$X = \#$ of white balls selected

$$P\{X=i\} = \frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}}$$

Suppose we number the m white balls $1, \ldots, m$

Define $X_i = \begin{cases} 1 & \text{if the } i\text{th white ball get selected} \\ 0 & \end{cases}$

$$X = X_1 + X_2 + \cdots + X_m$$

$$E[X_i] = P\{X_i = 1\} = P\{i\text{th white ball selected}\}$$

$$= \frac{\binom{1}{1}\binom{N-1}{n-1}}{\binom{N}{n}} = \frac{n}{N}$$

$$E[X] = \sum_{i=1}^{m} \frac{n}{N} = \frac{mn}{N}$$

$$Y_i = \begin{cases} 1 & \text{if } i\text{th ball selected is white} \\ 0 & \end{cases}$$

$$X = Y_1 + \cdots + Y_n$$

$$E[Y_i] = \frac{m}{N}$$

because when we consider all possible outcomes, ith ball is equally likely to be any of the N balls

$$E[X] = \frac{nm}{N}$$

(same as if selection is done w/ replacement)