Example of sum of independent discrete R.V.s

Exam 3 next friday
as before one sheet of notes is allowed.

If X and Y are independent R.V.s and discrete

$$p_{X,Y}(x,y) = p_X(x)\,p_Y(y)$$

$$p_{X+Y}(a) = \sum_y p_X(a-y)\,p_Y(y)$$

<u>Example</u> X poisson R.V. with parameter λ_1
Y poisson R.V. with parameter λ_2

Assume X and Y are independent: find p_{X+Y}

$$p_X(k) = e^{-\lambda_1}\frac{\lambda_1^k}{k!} \qquad p_Y(k) = e^{-\lambda_2}\frac{\lambda_2^k}{k!}$$

$$p_{X+Y}(n) = \sum_k p_X(n-k)\,p_Y(k)$$

$$= \sum_{k=0}^{n} e^{-\lambda_1}\frac{\lambda_1^{n-k}}{(n-k)!}\,e^{-\lambda_2}\frac{\lambda_2^k}{k!}$$

$p_X(n-k) > 0 \Rightarrow k \le n$

$p_Y(k) > 0 \Rightarrow 0 \le k$

$$= e^{-(\lambda_1+\lambda_2)} \sum_k \frac{\lambda_1^{n-k}}{(n-k)!} \frac{\lambda_2^{k}}{k!}$$

$$\left\{ \begin{aligned} (\lambda_1+\lambda_2)^n &= \sum_k \binom{n}{k} \lambda_1^{n-k} \lambda_2^{k} \\ &= \sum_k \frac{n!}{(n-k)!\,k!} \lambda_1^{n-k} \lambda_2^{k} \\ &= n! \sum_k \frac{\lambda_1^{n-k}}{(n-k)!} \frac{\lambda_2^{k}}{k!} \end{aligned} \right\}$$ Binomial Theorem

$$= e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1+\lambda_2)^n}{n!}$$

Conclusion: $X+Y$ has a Poisson distribution with parameter $\lambda_1+\lambda_2$

Next Conditional distribution:
(what happens when variables are not independent)

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

For discrete R.Vs X and Y

$$P\{X=x \mid Y=y\} = \frac{P\{X=x \text{ and } Y=y\}}{P\{Y=y\}}$$

We have a conditional probability mass function

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} \quad \left(= \frac{\text{joint PMF}}{\text{PMF of } Y} \right)$$

Condition distribution function

$$F_{X|Y}(a|y) = P\{X \le a \mid Y = y\} = \sum_{x \le a} p_{X|Y}(a|y)$$

X and Y are independent $\iff p_{X|Y}(x|y) = p_X(x)$

$$\left(\text{because} \quad p_{X|Y}(x|y) = \frac{p(x,y)}{p_Y(y)} = \frac{p_X(x)\, p_Y(y)}{p_Y(y)} = p_X(x) \right)$$

Example
- X is Poisson w/ parameter λ_1
- Y is Poisson w/ parameter λ_2

- Assume X and Y are independent
- We saw X+Y is Poisson w/ parameter $\lambda_1 + \lambda_2$

Compute conditional mass function of X given X+Y

$$P\{X = k \mid X+Y = n\} = \frac{P\{X = k,\ X+Y = n\}}{P\{X+Y = n\}}$$

$$= \frac{P\{X=k, Y=n-k\}}{P\{X+Y=n\}} = \frac{P\{X=k\}P\{Y=n-k\}}{P\{X+Y=n\}}$$

$$= \frac{\left(\cancel{e^{-\lambda_1}} \frac{\lambda_1^k}{k!}\right)\left(\cancel{e^{-\lambda_2}} \frac{\lambda_2^{n-k}}{(n-k)!}\right)}{\left[\cancel{e^{-(\lambda_1+\lambda_2)}} \frac{(\lambda_1+\lambda_2)^n}{n!}\right]}$$

$$= \frac{n!}{k!(n-k)!} \frac{\lambda_1^k \lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n} = \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

$$\left(p = \frac{\lambda_1}{\lambda_1+\lambda_2} \qquad 1-p = \frac{\lambda_2}{\lambda_1+\lambda_2}\right)$$

$$= \binom{n}{k} p^k (1-p)^{n-k}$$

Binomial w/ parameters n and $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$

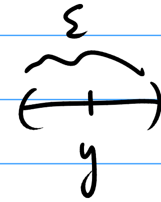
Continuous case: X and Y jointly continuous RVs

What if we try $P\{X \le a \mid Y=y\} = \frac{P\{X \le a \text{ and } Y=y\}}{P\{Y=y\}}$

Then have $P\{Y=y\} = \int_y^y f_Y(t)\,dt = 0$ We would divide by 0!

Can't divide by $P\{Y=y\}$ $\Rightarrow$ need different approach

Condition on $y - \frac{\varepsilon}{2} < Y < y + \frac{\varepsilon}{2}$



$$P\left\{y - \frac{\varepsilon}{2} < Y < y + \frac{\varepsilon}{2}\right\} = \int_{y-\frac{\varepsilon}{2}}^{y+\frac{\varepsilon}{2}} f_Y(t)\,dt \approx f_Y(y)\cdot\varepsilon$$

for small ε

$$P\left\{a < X < b \,\middle|\, y - \frac{\varepsilon}{2} < Y < y + \frac{\varepsilon}{2}\right\}$$

$$= \frac{P\left\{a < X < b \text{ and } y - \frac{\varepsilon}{2} < Y < y + \frac{\varepsilon}{2}\right\}}{P\left\{y - \frac{\varepsilon}{2} < Y < y + \frac{\varepsilon}{2}\right\}}$$

$$= \frac{\int_a^b \int_{y-\frac{\varepsilon}{2}}^{y+\frac{\varepsilon}{2}} f(x,t)\,dt\,dx}{\int_{y-\frac{\varepsilon}{2}}^{y+\frac{\varepsilon}{2}} f_Y(t)\,dt} \approx \frac{\int_a^b f(x,y)\,\varepsilon\,dx}{f_Y(y)\,\varepsilon}$$

$$= \int_a^b \left(\frac{f(x,y)}{f_Y(y)}\right) dx$$

The conditional density function of X given Y is $f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)}$

so $P\{a<X<b \mid Y=y\} = \int_a^b f_{X|Y}(x|y)\,dx$

Ex $f(x,y) = \begin{cases} \dfrac{e^{-x/y}e^{-y}}{y} & x>0 \text{ and } y>0 \\ 0 & \text{Otherwise} \end{cases}$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^{\infty} \frac{1}{y} e^{-x/y} e^{-y}\,dx$$

$$= e^{-y}$$

$$f_{X|Y}(x|y) = \frac{e^{-x/y}e^{-y}}{y} \Big/ e^{-y} = \frac{e^{-x/y}}{y}$$

So X given Y is exponentially distributed with parameter $\lambda = \frac{1}{y}$.