Next HW

Problems 6.18, 6.20, 6.29, 6.30, 6.32, 6.38, 6.42
Theoretical
Exercises: 6.11, 6.14

Sums of independent Random variables

Suppose X and Y are independent random variables
Suppose X and Y are continuous

density functions $f_X(x)$, $f_Y(y)$, joint $f_{X,Y}(x,y)$

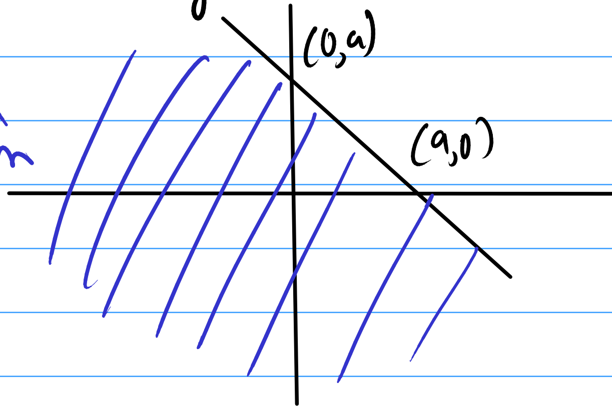
X and Y independent $\iff f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$

Now consider $X+Y$: find distribution & density

$$F_{X+Y}(a) = P\{X+Y \le a\}$$

$$= \iint_{\{x+y \le a\}} f_{X,Y}(x,y)\, dx\, dy$$

Region of integration



$x+y \le a$

$x \le a-y$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{a-y} f_{X,Y}(x,y)\,dx\,dy$$ So far, true for any R.Vs X, Y

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{a-y} f_X(x) f_Y(y)\,dx\,dy$$ using the fact that X and Y are independent

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{a-y} f_X(x)\,dx \right] f_Y(y)\,dy$$

$$= \int_{-\infty}^{\infty} F_X(a-y) f_Y(y)\,dy$$

$$\therefore \quad F_{X+Y}(a) = \int_{-\infty}^{\infty} F_X(a-y) f_Y(y)\,dy$$

$$f_{X+Y}(a) = \frac{d}{da} F_{X+Y} = \int_{-\infty}^{\infty} \frac{d}{da} F_X(a-y) f_Y(y)\,dy$$

$$= \int_{-\infty}^{\infty} f_X(a-y) f_Y(y)\,dy$$

Definition If f and g are two functions The convolution of f and g is new function

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy$$

Properties

(1) $f * g = g * f$

(2) $(f * g)' = (f' * g) = (f * g')$

To summarize if X and Y are independent R.V.s

then $F_{X+Y} = F_X * f_Y = f_X * F_Y$

$f_{X+Y} = f_X * f_Y$

Discrete case: integral replaced by sum

$$F_{X+Y}(a) = \sum_y F_X(a-y)\, p_Y(y)$$

$$= \sum_x p_X(x)\, F_Y(a-x)$$

$$p_{X+Y}(a) = \sum_y p_X(a-y)\, p_Y(y)$$

<u>Example</u> · Suppose X_1 is normal with mean μ_1 and variance σ_1^2

· Suppose X_2 is normal with mean μ_2 and variance σ_2^2

· Suppose X_1 and X_2 are independent

THEN $X_1 + X_2$ is normal with mean $\mu_1 + \mu_2$ and variance $\sigma^2 = \sigma_1^2 + \sigma_2^2$

ie. $f_{X_1+X_2} = f_{X_1} * f_{X_2}$ we're saying

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{(a-y-\mu_1)^2}{2\sigma_1^2}} \cdot \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}}\, dy$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2+\sigma_2^2}} e^{-\frac{(a-\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)}}$$

Straightforward but lengthy computation

Example Basketball team $P(\text{home win}) = .6$

$P(\text{away win}) = .5$

play 41 home and 41 away.

$X = \#$ home wins ~ binomial with $n=41$, $p=.6$

$Y = \#$ away wins ~ binomial with $n=41$, $p=.5$

X and Y are independent.

By DeMoivre-Laplace limit theorem

$X \sim$ normal w/ $\mu = np = 24.6$

$\sigma^2 = np(1-p) = 9.84$

$Y \sim$ normal w/ $\mu = np = 20.5$

$\sigma^2 = np(1-p) = 10.25$

Since "normal + normal = normal"

$X+Y \sim$ normal w/ $\mu = 24.6 + 20.5 = 45.1$

$\sigma^2 = 20.09$

P(50 or more wins in the entire season)

$= P(50 \leq X+Y) = P(49.5 < X+Y)$

$= P\left(\frac{49.5-45.1}{\sqrt{20.09}} < Z\right) = 1 - \Phi\left(\frac{49.5-45.1}{\sqrt{20.09}}\right)$

X and Y independent

X and Y are both exponential with rate λ

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x>0 \\ 0 & \end{cases} \qquad f_Y(y) = \begin{cases} \lambda e^{-\lambda y} & y>0 \\ 0 & \end{cases}$$

X and Y are "independent identically distributed"

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y) f_Y(y)\,dy$$

in order for $f_X(a-y) f_Y(y)$ to be nonzero

we need $\quad a-y>0 \Rightarrow y<a$
$\qquad\qquad y>0$

$$= \int_0^a \lambda e^{-\lambda(a-y)} \lambda e^{-\lambda y}\,dy$$

$$= \lambda^2 e^{-\lambda a} \int_0^a \underbrace{\cancel{e^{+\lambda y}} \cancel{e^{-\lambda y}}}_{1} dy$$

$$= \lambda^2 e^{-\lambda a} a = \lambda e^{-\lambda a} \frac{(\lambda a)^1}{\Gamma(2)}$$

Observe that X+Y is a Gamma variable with rate λ and parameter $\alpha = 2$

$$\left(\begin{array}{c}\text{Gamma} \\ \lambda, \alpha\end{array} \quad f(x) = \lambda e^{-\lambda x} \frac{(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} \right)$$

Observe that exponential RV. = Gamma random variable with $\alpha = 1$

in general if X is Gamma w/ λ, α
Y is Gamma w/ λ, β

and X and Y are independent

then $X+Y$ is Gamma w/ $\lambda, \alpha+\beta$

Conceptual explanation: if α and β are integers

then X = Gamma w/ λ, α = time to wait for α events in Poisson process w/ rate λ

Y = Gamma w/ λ, β = time to wait for β events

$X+Y$ = time to wait for $\alpha+\beta$ events

= Gamma w/ $\lambda, \alpha+\beta$