Independent Random Variables

Recall: given a pair X, Y of random vars.
consider joint distribution

$$F(a,b) = P\{X \le a, Y \le b\}$$

Discrete case: $p(x,y) = P\{X = x, Y = y\}$

Continuous: $f(x,y)$

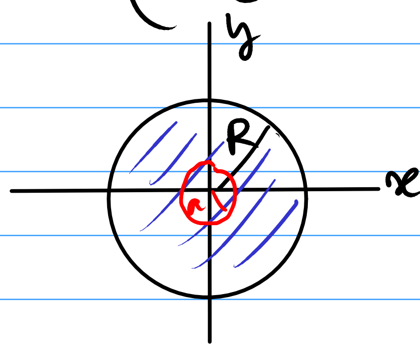
$$P\{(X,Y) \in C\} = \iint_C f(x,y)\,dx\,dy$$

Example of jointly continuous R.V.s

Define joint density function of X and Y to

$$f(x,y) = \begin{cases} c & \text{if } x^2 + y^2 \le R^2 \\ 0 & \text{otherwise} \end{cases}$$

Picture



f is nonzero on disk of radius R.

find C: $1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy$

$$= \iint_{\{x^2+y^2 \le R^2\}} C\,dx\,dy + \cancel{\iint_{\{x^2+y^2 > R^2\}} 0\,dx\,dy}$$

$$= C \iint_{\{x^2+y^2 \le R^2\}} 1\,dx\,dy = \int_{-R}^{R}\int_{-\sqrt{R^2-y^2}}^{+\sqrt{R^2-y^2}} 1\,dx\,dy$$

$$= C\,\text{Area}\left(\{x^2+y^2 \le R^2\}\right) = C\pi R^2 = 1$$

$$C = \frac{1}{\pi R^2}$$

Q. What is prob. that a randomly chosen point lies at a distance $\le a$ from the center?

$$P(X^2+Y^2 \le a^2\} = C \iint_{\{x^2+y^2 \le a^2\}} 1\,dx\,dy = C\,\text{Area}(\ )$$

$$= C \cdot \pi a^2 = \frac{\pi a^2}{\pi R^2} = \frac{a^2}{R^2}$$

Note added: This is only true if $a \le R$

if $a > R$, then $P\{X^2+Y^2 \le a^2\} = 1$

Recall A and B are independent events

if $P(A \cap B) = P(A)P(B)$

X and Y are independent random variables

if, for any two sets C and D of real numbers

$P((X \in C) \text{ and } (Y \in D)) = P(X \in C) \cdot P(Y \in D)$

i.e., $X \in C$ and $Y \in D$ are independent events.

If X and Y are independent

- $P\{a_1 < X \leq a_2, b_1 < Y \leq b_2\}$

  $= P\{a_1 < X \leq a_2\} \cdot P\{b_1 < Y \leq b_2\}$

- $P\{X \leq a, Y \leq b\} = P\{X \leq a\} \cdot P\{Y \leq b\}$

i.e. $F_{X,Y}(a,b) = F_X(a) \cdot F_Y(b)$

In fact X and Y are independent if

$F_{X,Y}(a,b)$ is the product of $F_X(a)$ and $F_Y(b)$

- In discrete case: joint prob. mass function

  $p(x,y) = P\{X=x, Y=y\} = P\{X=x\} \cdot P\{Y=y\}$

(*) $p(x,y) = p_X(x)\, p_Y(y)$

In fact (*) is equivalent to X and Y being indep.

$$P\{X \in C, Y \in D\} = \sum_{y \in D} \sum_{x \in C} p(x,y)$$

$$= \sum_{y \in D} \sum_{x \in C} p_X(x) p_Y(y) \quad \text{(assumption.)}$$

$$= \sum_{y \in D} \left(p_Y(y) \sum_{x \in C} p_X(x) \right)$$

$$= \left(\sum_{x \in C} p_X(x) \right) \left(\sum_{y \in D} p_Y(y) \right)$$

$$= P\{X \in C\} \cdot P\{Y \in D\}$$

Thus X and Y are independent $\boxtimes$

For Continuous Random Variables X, Y

X and Y are independent $\Leftrightarrow f_{X,Y}(x,y) = f_X(x) f_Y(y)$

$\Leftrightarrow f_{X,Y}(x,y) = h(x) g(y)$ for some functions $h(x), g(y)$

Discrete example: $n+m$ Bernoulli trials w/ prob p of success

X = # success in first n trials

Y = # success in last m trials

X and Y are independent

$$P\{X=i, Y=j\} = P\{X=i\}P\{Y=j\}$$
$$= \binom{n}{i} p^i (1-p)^{n-i} \cdot \binom{m}{j} p^j (1-p)^{m-j}$$

Z = # of successes in all $n+m$ trials ($Z = X+Y$)

X and Z are not independent

$$P\{X=i, Z=i+j\} = P\{X=i, Y=j\} = P\{X=i\}P\{Y=j\}$$

$$P\{X=i\} \cdot P\{Z=i+j\} \neq P\{X=i\}P\{Y=j\}$$

Continuous example Alice and Bob are to meet at certain location

each arrives at a time between 12 and 1 uniformly in this interval, and independently of each other.

X = time in minutes after 12 that Alice arrives

Y = time in minutes " " " Bob "

Q Prob. that Alice arrives first?

A:

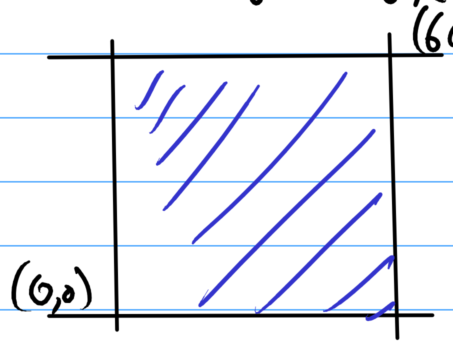
$$P\{X < Y\} = \frac{1}{2} \quad \text{(intuitive by symmetry)}$$

$$f_X(x) = \begin{cases} \frac{1}{60} & 0 < x < 60 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{60} & 0 < y < 60 \\ 0 & \text{otherwise} \end{cases}$$

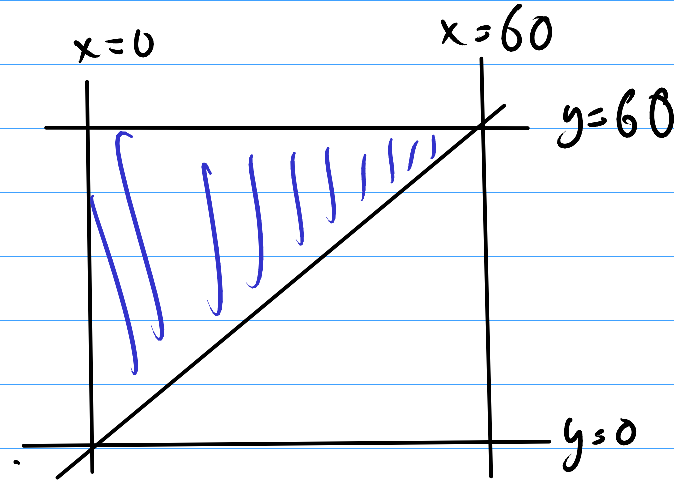
By independence

$$f(x,y) = f_X(x) f_Y(y) = \begin{cases} \frac{1}{(60)^2} & 0 < x < 60 \text{ and } 0 < y < 60 \\ 0 & \text{otherwise} \end{cases}$$



$$P\{X<Y\} = \iint_{\{x<y\}} f(x,y)\,dx\,dy$$

$$= \iint_{\left\{\substack{0<x<60 \\ 0<y<60 \\ x<y}\right\}} \frac{1}{(60)^2}\,dx\,dy$$



$$= \int_0^{60} \int_0^{y} \frac{1}{(60)^2}\,dx\,dy$$

$$= \frac{1}{(60)^2} \int_0^{60} y\,dy = \frac{1}{(60)^2}\,\frac{1}{2}\,(60)^2 = \frac{1}{2}$$