Distribution of a function of a R.V
and joint distributions.

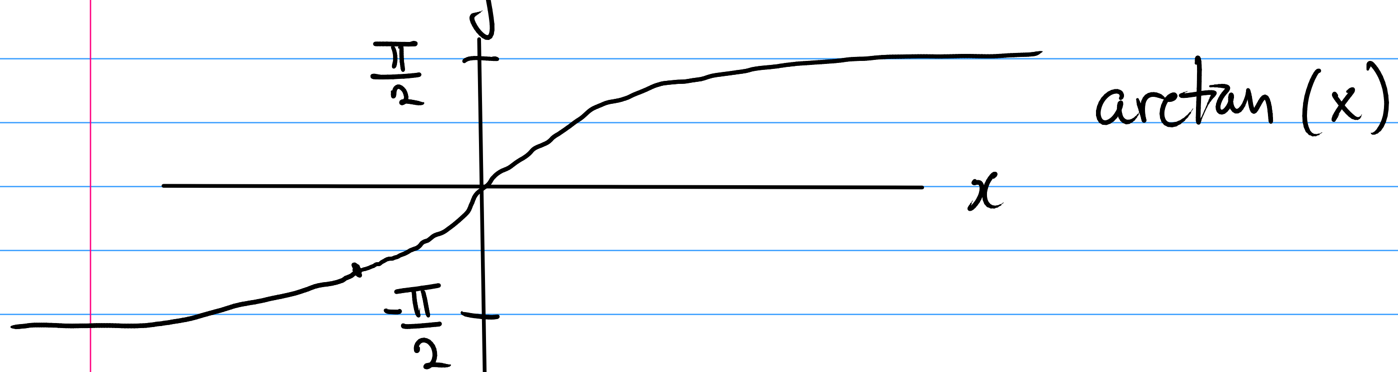
Suppose X and Y are continuous random variables

Suppose $Y = g(X)$ $g(x)$ is a function

Relationship between density/distribution functions of X and Y.

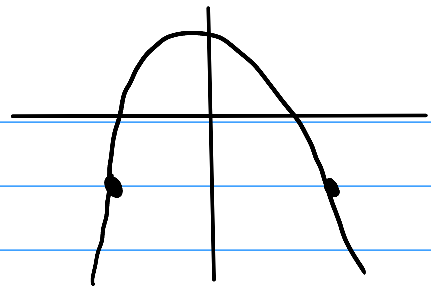
Special Case g is an increasing function

Assume $g'(x) > 0$ for all x



Two properties range of g is some interval (a,b)
a could be $-\infty$
b could be $+\infty$

for each $y \in (a,b)$, there is a unique x such that
$$g(x) = y \Leftrightarrow g^{-1}(y) = x$$



impossible b/c $g'(x) > 0$

$$F_Y(y) = P\{Y \le y\} = P\{g(X) \le y\}$$

if y is in range of $g = (a, b)$

$$= P\{X \le g^{-1}(y)\}$$

inequality preserved b/c g increasing.

$$= F_X(g^{-1}(y))$$

if $y < a$ so is below range of g

$$F_Y(y) = P\{g(X) \le y\} = 0$$

if $y > b$ so y is above range of g

$$F_Y(y) = P\{g(X) \le y\} = 1$$

$$F_Y(y) = \begin{cases} 0 & y \le a \\ F_X(g^{-1}(y)) & a < y < b \\ 1 & b \le y \end{cases}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$(\text{if } a<y<b) = \frac{d}{dy} F_X(g^{-1}(y)) = F_X'(g^{-1}(y))(g^{-1})'(y)$$

$$= f_X(g^{-1}(y))(g^{-1})'(y)$$

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y))(g^{-1})'(y) & a<y<b \\ 0 & \text{otherwise} \end{cases}$$

$Y = \arctan(X)$ $\quad (a,b) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$g = \arctan$ $\quad$ Range

$g^{-1} = \tan$ $\quad (g^{-1})' = \sec^2$

$$f_Y(y) = f_X(\tan(y)) \sec^2 y \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

This is probability analog of change-of-variables in an integral (u substitution)

Probability with 2 random variables

X, Y 2 random variables

Joint distribution function

$$F(a,b) = P\{X \le a, Y \le b\}$$

encodes info about probabilities of X, of Y and of events defined in terms of X and Y.

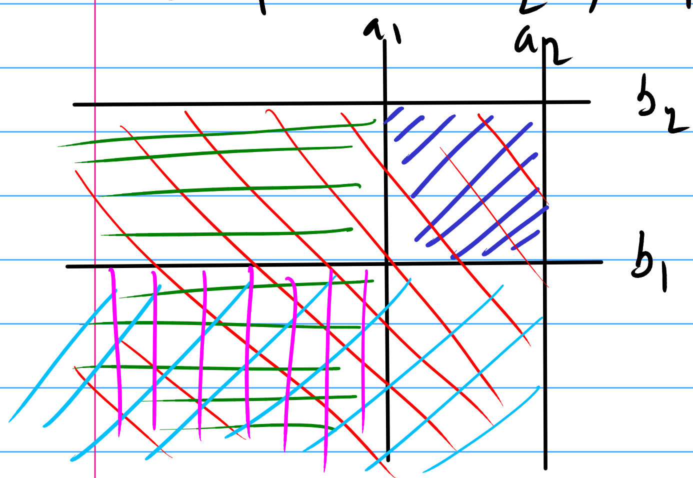
$$F_X(a) = P\{X \le a\} = P\{X \le a, Y < \infty\}$$

$$= \lim_{b \to \infty} F(a,b)$$

$$\equiv F(a,\infty)$$

$$F_Y(b) = \lim_{a \to \infty} F(a,b) \equiv F(\infty, b)$$

$$P\{a_1 < X \le a_2,\ b_1 < Y \le b_2\} =$$



$$F(a_2,b_2)$$
$$-F(a_1,b_2)$$
$$-F(a_2,b_1)$$
$$+F(a_1,b_1)$$

For X and Y discrete R.V.'s
joint probability mass function

$$p(x,y) = P\{X=x, Y=y\}$$

C = some set of pairs of possible values

$$P\{(X,Y) \in C\} = \sum_{(x,y)\in C} p(x,y)$$

$$p_X(x) = P\{X=x\} = P\{X=x, Y=\text{anything}\}$$

$$= \sum_{\substack{y \\ \text{possible} \\ \text{values of } Y}} p(x,y)$$

$$p_Y(y) = \sum_{x} p(x,y)$$

$$F(a,b) = P\{X \le a, Y \le b\} = \sum_{x \le a} \sum_{y \le b} p(x,y)$$

Roll two 4-sided dice

X = first die (1, 2, 3, 4)

Y = second die (1, 2, 3, 4)

Z = sum (2, 3, 4, 5, 6, 7, 8)

Joint PMF of X and Y

x \ y	1	2	3	4	p_X
1	1/16	1/16	1/16	1/16	1/4
2	1/16	1/16	1/16	1/16	1/4
3	1/16	1/16	1/16	1/16	1/4
4	1/16	1/16	1/16	1/16	1/4
p_Y	1/4	1/4	1/4	1/4	

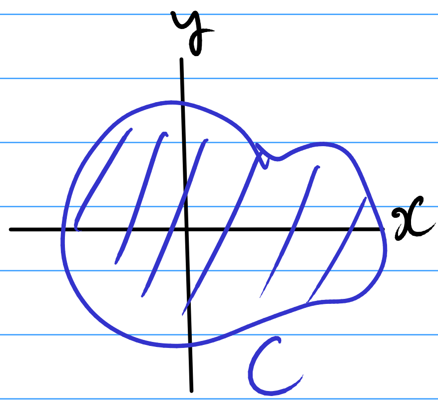
Joint PMF of X and Z

x \ z	2	3	4	5	6	7	8	p_X
1	1/16	1/16	1/16	1/16	0	0	0	1/4
2	0	1/16	1/16	1/16	1/16	0	0	1/4
3	0	0	1/16	1/16	1/16	1/16	0	1/4
4	0	0	0	1/16	1/16	1/16	1/16	1/4
p_Z	1/16	2/16	3/16	4/16	3/16	2/16	1/16	

Continuous case: X and Y are jointly continuous if there is a density function

$f(x,y)$ such that

$$P\{(X,Y) \in C\} = \iint_C f(x,y)\,dx\,dy$$



In particular $P\{a_1 < X \le a_2, b_1 < Y \le b_2\} = \int_{b_1}^{b_2}\int_{a_1}^{a_2} f(x,y)\,dx\,dy$

$$F(a,b) = \int_{-\infty}^{b}\int_{-\infty}^{a} f(x,y)\,dx\,dy$$

conversely $f(a,b) = \dfrac{\partial^2}{\partial a\,\partial b} F(a,b)$