Combinatorial Proof and the Binomial Theorem

Combinatorial Proof = Proof technique
count something in two different ways,
get two different formulas, which must
then be equal.

Recall $\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$ = # subsets of size r in a set of total size n.

Theorem

$$\binom{n}{r} = \boxed{\binom{n-1}{r-1} + \binom{n-1}{r}} \quad (1 \le r \le n)$$

Proof Right hand side

$$\frac{(n-1)!}{((n-1)-(r-1))!\,(r-1)!} + \frac{(n-1)!}{(n-1-r)!\,r!}$$

$$\frac{r}{r} \cdot \frac{(n-1)!}{(n-r)!\,(r-1)!} + \frac{(n-1)!}{(n-r-1)!\,r!} \cdot \frac{(n-r)}{(n-r)}$$

$$= \frac{r \cdot (n-1)!}{(n-r)!\ r!} + \frac{(n-1)! \cdot (n-r)}{(n-r)!\ r!}$$

using $r! = r \cdot (r-1)!$
$(n-r)! = (n-r) \cdot (n-r-1)!$

$$= \frac{r\,(n-1)! + (n-1)!\,(n-r)}{(n-r)!\ r!}$$

$$= \frac{n\,(n-1)!}{(n-r)!\,r!} = \frac{n!}{(n-r)!\,r!} = \binom{n}{r} \text{ QED.}$$

Combinatorial Proof is "conceptual"

Q: $S = \{1, 2, \dots, n\}$

How many subsets of size r?

A1. $\binom{n}{r}$

A2. # subsets of size r

= # subsets of size r that contain 1

+ # subsets of size r that don't contain 1

subsets of size r containing 1

choose r−1 elements from $\underbrace{\{2, 3, \ldots, n\}}_{n-1 \text{ of these}}$

$\binom{n-1}{r-1}$

subsets of size r not containing 1

choose r elements from $\{2, 3, \ldots, n\}$

$\binom{n-1}{r}$

subsets of size $r = \binom{n-1}{r-1} + \binom{n-1}{r}$

Therefore $\underbrace{\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}}_{\text{Pascal's Triangle identity}}$

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\vdots \qquad \vdots$$

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Left-right symmetry $\Leftrightarrow \binom{n}{r} = \binom{n}{n-r}$

The Binomial Theorem is a bit of abstract algebra.

x, y variables $\quad x^5y^2$ – monomial

$x^4 + y^3$ – binomial

$x + x^2 + xy + y^2$ – polynomial

Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$(x+y)^3 = \binom{3}{0}x^0y^3 + \binom{3}{1}x^1y^2 + \binom{3}{2}x^2y^1 + \binom{3}{3}x^3y^0$$

$= 1y^3 + 3xy^2 + 3x^2y + 1x^3$

a row of pascal's triangle

2 Proofs

One proof uses induction

Another proof use combinatorics

Combinatorics Proof

$(x+y)^2 = (x+y)(x+y) = x\cdot x + x\cdot y + y\cdot x + y\cdot y$

$(x+y)^3 = (x+y)(x\cdot x + x\cdot y + y\cdot x + y\cdot y)$

8 terms

$= x\cdot x\cdot x + x\cdot x\cdot y + x\cdot y\cdot x + x\cdot y\cdot y$

$+ y\cdot x\cdot x + y\cdot x\cdot y + y\cdot y\cdot x + y\cdot y\cdot y$

$= x^3 + 3x^2y + 3xy^2 + y^3$

$$(x+y)^n = \underbrace{(x+y)(x+y)(x+y)\cdots(x+y)}_{n \text{ factors}}$$

Expand this out keep in track of the order in which the factors appear

To get a term can do, for example

Take x from the first factor
Take x from the second factor
Take y from the third factor
⋮
Take y from the nth factor

Gives a term $x \cdot x \cdot y \cdots y$

Total number of terms 2^n:

How many have k x's and n-k y's

Each way of arranging k x's and n-k y's will appear exactly once as a term

Permutations of n objects, k indistinguishable, n-k indistinguishable

$$\frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$ = # of times you get k x's and $n-k$ y's

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$ QED.