Exponential & Gamma Random Variables

HW Problems 5.32, 5.34, 5.39
Theoretical 5.13, 5.30, 5.31

Ch6 Problems 6.1, 6.7, 6.9

Sequel to lecture on Poisson process

X = exponential R.V. with rate parameter λ has density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

(So X is a nonnegative R.V.)

Cumulative distribution function

$$F_X(a) = \int_0^a \lambda e^{-\lambda x} dx = \left[-e^{-\lambda x}\right]_0^a$$

$$= 1 - e^{-\lambda a} \quad (\text{if } a \geq 0)$$

and $F_X(a) = 0$ if $a < 0$

Compute $E[X^n] = \int_{-\infty}^{\infty} x^n f(x) dx$

$$= \int_0^{\infty} x^n f(x) dx = \int_0^{\infty} x^n \lambda e^{-\lambda x} dx$$

integration by parts $\quad u = x^n \quad du = nx^{n-1}$

$dv = \lambda e^{-\lambda x} \quad v = -e^{-\lambda x}$

$$\int_0^\infty x^n \lambda e^{-\lambda x} dx = \left[x^n(-e^{-\lambda x})\right]_0^\infty - \int_0^\infty nx^{n-1}(-e^{-\lambda x})\,dx$$

$$= (0 - 0) + \int_0^\infty nx^{n-1} e^{-\lambda x} dx$$

$$= \frac{n}{\lambda} \int_0^\infty x^{n-1} \lambda e^{-\lambda x} dx$$

$$= \frac{n}{\lambda} E[X^{n-1}]$$

$$\therefore \quad E[X^n] = \frac{n}{\lambda} E[X^{n-1}]$$

$$E[X] = E[X^1] = \frac{1}{\lambda} E[X^0] = \frac{1}{\lambda}$$

$$E[X^2] = \frac{2}{\lambda} E[X] = \frac{2}{\lambda}\frac{1}{\lambda} = \frac{2}{\lambda^2}$$

$$Var(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

Recall Poisson process: events are happening randomly, but with average rate λ.

$N(t)$ = # of events in interval of time of length t.

$$P\{N(t)=k\} = e^{-\lambda t}\frac{(\lambda t)^k}{k!} \quad \text{Poisson dist.}$$

X = time we have to wait until the first event happens.

X is then an exponential random variable with rate parameter λ.

<u>Proof</u> $P\{X>t\} = P\{N(t)=0\} = e^{-\lambda t}$

$$F_X(t) = P\{X \le t\} = 1 - P\{X>t\} = 1 - e^{-\lambda t}$$

$$F_X(t) = 1 - e^{-\lambda t}$$ Therefore X is exponential.

<u>Ex</u> Suppose open store, X = time for first customer to arrive. X is exponential with $\lambda = 10 \frac{\text{customers}}{\text{hour}}$.

$$P\{X > .2 \text{ hours}\} = e^{-\lambda(.2 \text{ hours})} = e^{-2}$$

$$E[X] = \frac{1}{\lambda} = \frac{1}{10} \text{ hour.}$$

Special property of exponential dist:
MEMORYLESSNESS

$$P\{X > s+t \mid X > t\} = P\{X > s\}$$

$$\frac{P\{X > s+t\}}{P\{X > t\}} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} = \frac{e^{-\lambda s} e^{-\lambda t}}{e^{-\lambda t}} = e^{-\lambda s} = P\{X > s\}$$

(Exp dist is only continuous distribution with this property)

(geometric R.V. is discrete distribution with this property)

If X is time to wait for an event, it doesn't matter how long we already waited:

Ex Waiting to be served by customer service

$E[X] = 5 \text{ min} = \frac{1}{\lambda}$ $\lambda = .2$ per minute

$P\{X > 5 \text{ min}\} = e^{-.2 \cdot 5} = e^{-1} = .368$

$P\{X > 10 \text{ min} \mid X > 5 \text{ min}\} = P\{X > 5 \text{ min}\} = e^{-1}$

$P\{X > 65 \text{ min} \mid X > 60 \text{ min}\} = P\{X > 5 \text{ min}\} = e^{-1}$

→ It doesn't matter when we start waiting

→ In Poisson time between two events or time from any point until next event is also exponentially distributed with same rate λ

Analogy

Bernoulli	Poisson
Binomial	Poisson
geometric	exponential
negative binomial	Gamma distribution

One thing Gamma distribution models is the amount of time we must wait for k events to occur in the poisson process.

Y_k = Gamma RV with parameters λ, k

$$f_{Y_k}(t) = \begin{cases} \dfrac{\lambda e^{-\lambda t}(\lambda t)^{k-1}}{(k-1)!} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

We can extend the definition to allow k (integer) to become α (real number)

Y_α = Gamma RV w/ parameters $\lambda, \alpha > 0$

$$f_{Y_\alpha}(x) = \begin{cases} \lambda e^{-\lambda x} \dfrac{(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Since $\int_0^\infty f_{Y_\alpha}(x)\,dx = 1 \Rightarrow \underbrace{\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1}\,dx}_{\text{This is definition of } \Gamma\text{-function}}$

$$\Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1), \quad \Gamma(1) = 1$$

$$\Gamma(k+1) = k!$$

Gamma dist with $\lambda = \frac{1}{2}$, $\alpha = \frac{n}{2}$

this is called χ_n^2 - distribution