Normal variables and DeMoivre–Laplace Limit theorem

X normal R.V. with mean μ variance σ^2
has PDF

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

Z standard normal R.V. (has mean 0, variance 1)
has PDF

$$f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

CDF of Z is called Φ

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\,dy$$

This is a special function

Q: What about CDF of X w/ μ, σ

$$F_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{x} e^{-(x-\mu)^2/2\sigma^2}\,dx$$

A: $F_X(x)$ may be computed in terms of Φ

For any random variable X with

$$E[X] = \mu \quad \text{and} \quad Var(X) = \sigma^2$$

Define $Z = \frac{X-\mu}{\sigma}$

$E[Z] = 0$

$$E[Z] = E\left[\frac{X-\mu}{\sigma}\right] = \frac{E[X]-\mu}{\sigma} = 0$$

$Var(Z) = 1$

$$Var(Z) = E[Z^2] = E\left[\frac{(X-\mu)^2}{\sigma^2}\right] = \frac{1}{\sigma^2} E[(X-\mu)^2]$$

$$= \frac{1}{\sigma^2} Var(X) = 1$$

Conversely if Z has $E[Z]=0$ and $Var(Z)=1$

the $X = \sigma Z + \mu$ is a random variable

with $E[X] = \mu$ $Var(X) = \sigma^2$

If X is any random variable then

$Z = \frac{X-\mu}{\sigma}$ will be called the standardization of X.

If Z is a standard normal random variable (with mean 0 and variance 1)

then $X = \sigma Z + \mu$ is a normal random variable with mean μ and variance σ^2

Conversely: if X is normal with mean μ and variance σ^2, then

$$Z = \frac{X-\mu}{\sigma}$$ is normal mean 0 variance 1

Proof Z normal $\Rightarrow$ X normal

CDF of $X = F_X(x) = P\{X \leq x\}$

$$= P\{\sigma Z + \mu \leq x\}$$

$$= P\left\{Z \leq \frac{x-\mu}{\sigma}\right\}$$

$$= F_Z\left(\frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

So $F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$ (very useful)

Density function of X $f_X(x)$

$$f_X(x) = \frac{d}{dx} F_X(x) = \Phi'\left(\frac{x-\mu}{\sigma}\right) \cdot \frac{1}{\sigma}$$

$$\Phi'(y) = f_Z(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2/2} \cdot \frac{1}{\sigma}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

So X is normal with mean μ and variance σ^2

Computing probabilities for normal variables with $\Phi(x)$:

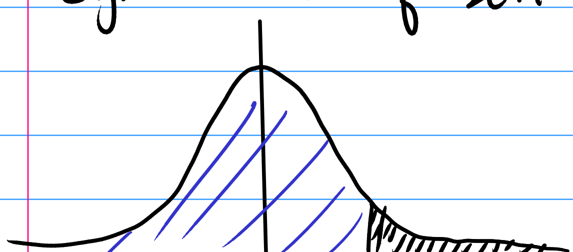
For any variable X with PDF f_X, CDF F_X

$$P\{a \le X \le b\} = \int_a^b f_X(x)\,dx = F_X(b) - F_X(a)$$

Z standard normal R.V.

$$P\{a \le Z \le b\} = \Phi(b) - \Phi(a)$$

Symmetries of bell curve:



$\Phi(x)$ $\qquad x \qquad P\{Z>x\} = P\{Z<-x\} = \Phi(-x)$

$P\{Z>x\} = 1 - \Phi(x)$

Also $\Phi(-x) = 1 - \Phi(x)$

(Table of values on p. 201 has $\Phi(x)$ for $x > 0$. Use this relation to find $\Phi(x)$ for $x < 0$)

<u>Ex</u> X normal w/ $\mu = 3$, $\sigma^2 = 9$, $\sigma = 3$

Q $P\{2 < X < 5\} = P\left\{\frac{2-3}{3} < \frac{X-3}{3} < \frac{5-3}{3}\right\}$

$= P\{-\frac{1}{3} < Z < \frac{2}{3}\}$

$= \Phi(\frac{2}{3}) - \Phi(-\frac{1}{3})$

$= \Phi(\frac{2}{3}) - [1 - \Phi(\frac{1}{3})]$

$= \Phi(\frac{2}{3}) + \Phi(\frac{1}{3}) - 1 = .7454 + .6293 - 1$

$= .3779$

<u>Q</u>: if X is normal w/ μ, σ

$P\{\mu < X < \mu + \sigma\} = P\left\{\frac{\mu - \mu}{\sigma} < Z < \frac{\mu + \sigma - \mu}{\sigma}\right\}$

$= P\{0 < Z < 1\} = \Phi(1) - \Phi(0)$

$= .3413$

DeMoivre - Laplace limit theorem
(Approximating binomial R.V. by Normal R.V.)

S_n = binomial w/ parameters n, p
(S_n = # of success in n trials
$P(\text{success}) = p$.)

Mean $= E[S_n] = np$ $\quad Var(S_n) = np(1-p)$

Standardize

$$\frac{S_n - np}{\sqrt{np(1-p)}}$$ has mean 0 and verince 1

DeMoive & Laplace say

$$\frac{S_n - np}{\sqrt{np(1-p)}} \sim \text{Standard normal } (\mu = 0 \quad \sigma = 1) \quad \text{as } n \to \infty$$

if n large

$$P\left\{a \le \frac{S_n - np}{\sqrt{np(1-p)}} \le b\right\} \approx P\{a \le Z \le b\} = \Phi(b) - \Phi(a)$$

As a rule approximation is good if $np(1-p) \ge 10$

Example Fair $(p = .5)$ coin is flipped 40

$X = \#$ heads

Use normal approx to compute $P(X=20)$

Wrong: $P\{X=20\} = P\left\{ \frac{X-20}{\sqrt{10}} = \frac{20-20}{\sqrt{10}} \right) = P(Z=0)$

$= \Phi(0) - \Phi(0) = 0$!

CONTINUITY = use half-integers in inequality

$P(X=20) = P\{19.5 < X < 20.5\}$

$= P\left\{ \frac{19.5-20}{\sqrt{10}} < Z < \frac{20.5-20}{\sqrt{10}} \right\}$

$= P\{-.16 < Z < .16\}$

$\approx .1272$

exact $\binom{40}{20} \left(\frac{1}{2}\right)^{40} = .1254$