Uniform and Normal Random Variables

Note: Theoretical Exercise 5.2, NOT 5.1 is on Homework

* Q-drop deadline is next Monday, April 2.

X is uniform on (α, β) if it has PDF

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \alpha < x < \beta \\ 0 & \text{otherwise} \end{cases}$$

Cumulative distribution function CDF

$$F(a) = \int_{-\infty}^{a} f(x)\,dx$$

if $a \leq \alpha$ $\quad \int_{-\infty}^{a} f(x)\,dx = \int_{-\infty}^{a} 0\,dx = 0$

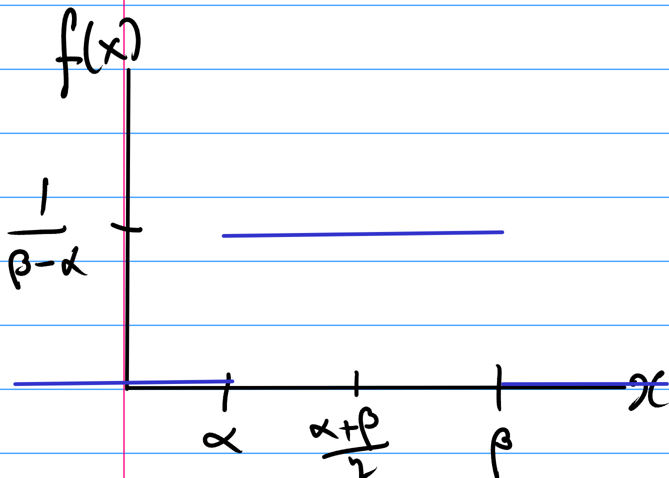
If $\alpha < a < \beta$

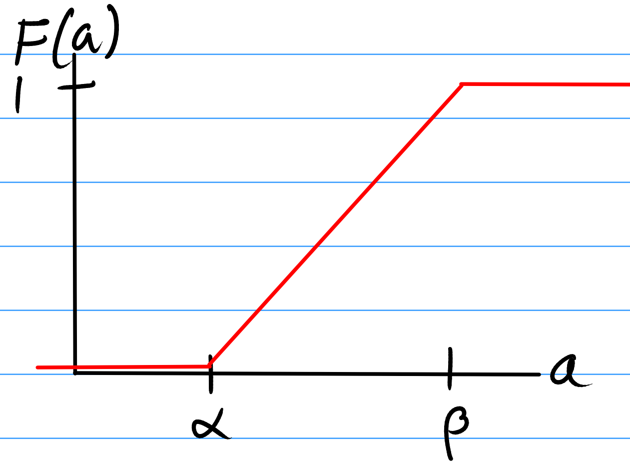
$$\int_{-\infty}^{a} f(x)\,dx = \int_{\alpha}^{a} f(x)\,dx = \int_{\alpha}^{a} \frac{1}{\beta-\alpha}\,dx$$

$$= \left[\frac{1}{\beta-\alpha} x\right]_{x=\alpha}^{x=a} = \frac{a-\alpha}{\beta-\alpha}$$

If $\beta \leq a$ $\quad \int_{-\infty}^{a} f(x)\,dx = \int_{\alpha}^{\beta} \frac{1}{\beta-\alpha}\,dx = \frac{\beta-\alpha}{\beta-\alpha} = 1$

$$F(a) = \begin{cases} 0 & a \le \alpha \\ \dfrac{a-\alpha}{\beta-\alpha} & \alpha < a < \beta \\ 1 & \beta \le a \end{cases}$$





$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{\alpha}^{\beta} x \frac{1}{\beta-\alpha}\,dx$$

$$= \frac{1}{\beta-\alpha}\left[\frac{x^2}{2}\right]_{\alpha}^{\beta} = \frac{1}{\beta-\alpha}\frac{1}{2}(\beta^2-\alpha^2)$$

$$= \frac{1}{\cancel{\beta-\alpha}}\frac{1}{2}\cancel{(\beta-\alpha)}(\beta+\alpha) = \frac{1}{2}(\beta+\alpha)$$

$= \text{midpoint of } (\alpha, \beta)$

Variance: $E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{\alpha}^{\beta} x^2 \frac{1}{\beta-\alpha}\,dx$

$$= \frac{1}{\beta-\alpha}\left[\frac{x^3}{3}\right]_{\alpha}^{\beta} = \frac{1}{\beta-\alpha}\frac{1}{3}(\beta^3-\alpha^3)$$

Fact $(\beta^3 - \alpha^3) = (\beta - \alpha)(\beta^2 + \alpha\beta + \alpha^2)$

$$E[X^2] = \frac{1}{3}(\beta^2 + \alpha\beta + \alpha^2)$$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$

$$= \frac{1}{3}(\beta^2 + \alpha\beta + \alpha^2) - \left(\frac{1}{2}(\beta + \alpha)\right)^2$$

$$= \frac{1}{3}(\beta^2 + \alpha\beta + \alpha^2) - \frac{1}{4}(\beta^2 + 2\alpha\beta + \alpha^2)$$

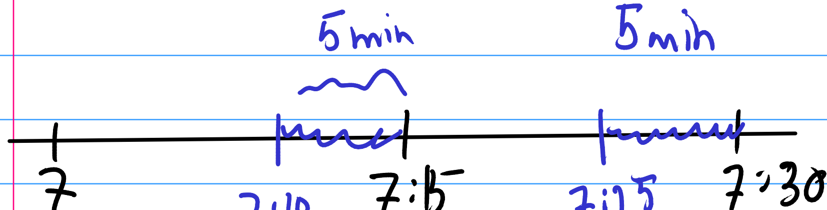
$$= \frac{1}{12}(\beta^2 - 2\alpha\beta + \alpha^2) = \frac{1}{12}(\beta - \alpha)^2$$

Observe $\beta - \alpha$ = width of (α, β)

<u>Example</u> Busses arrive at a stop at 7, 7:15, 7:30, ...

Passenger arrives at a time uniformly distributed in the interval (7:00, 7:30)

P(wait less than 5 minutes)



5 + 5 minutes out of 30 minutes

$$P = \frac{10}{30} = \frac{1}{3}$$

Normal Random Variable

a very "universal" distribution

→ approximates binomial distribution
(DeMoivre - Laplace)

→ model errors in scientific observations (Gauss)

→ Models any sum of independent identically distributed RVs with finite mean and variance

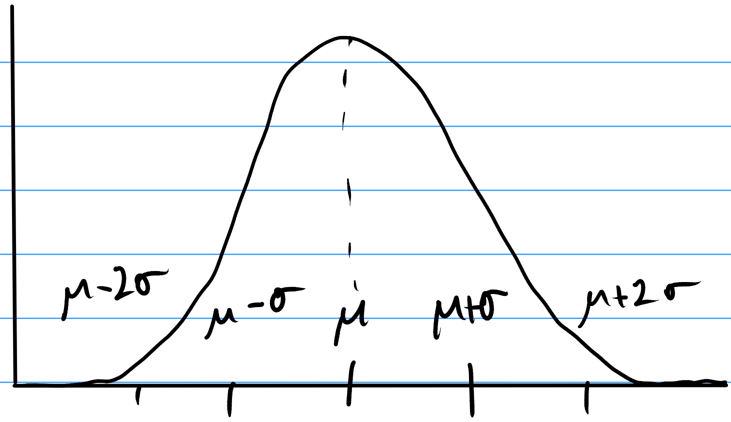
(Central Limit Theorem)

Normal Random Variable with mean μ & variance σ^2
(standard deviation σ)

has PDF

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2} \qquad (-\infty < x < \infty)$$

Bell curve



Why is $\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}\,dx = 1$?

$$\parallel$$

sub $y = \frac{x-\mu}{\sigma}$ $\qquad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2/2}\,dy$

$dy = \frac{dx}{\sigma}$

Need to show $I = \int_{-\infty}^{\infty} e^{-y^2/2}\,dy = \sqrt{2\pi}$

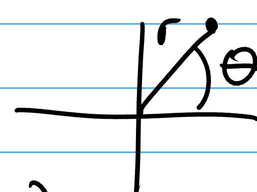
Look at $I^2 = \left(\int_{-\infty}^{\infty} e^{-y^2/2}\,dy\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2}\,dy\right)$

$$= \left(\int_{-\infty}^{\infty} e^{-y^2/2}\,dy\right)\left(\int_{-\infty}^{\infty} e^{-x^2/2}\,dx\right)$$

$$\overset{\text{shuffle}}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-y^2/2}\, e^{-x^2/2}\,dy\,dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\,dy\,dx$$

change to polar coordinates



$r^2 = x^2 + y^2 \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$dy\,dx = r\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2/2}\, r\,dr\,d\theta$$

$$\int_0^{\infty} r e^{-r^2/2}\,dr \qquad u = r^2/2 \qquad du = r\,dr$$

$$= \int_0^{\infty} e^{-u}\,du = \left[-e^{-u}\right]_{u=0}^{u=\infty} = 0 - (-e^{-0}) = 1$$

$$I^2 = \int_0^{2\pi} 1\,d\theta = 2\pi \Rightarrow I = \sqrt{2\pi} \qquad \blacksquare$$

<u>Q</u>: Why we just find the antiderivative of $e^{-y^2/2}$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy$$

then by FTC, $\frac{d}{dx}\Phi = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

<u>Fact</u>: $\Phi(x)$ cannot be expressed in terms of "elementary functions" (Liouville)

elementary functions:

$$\frac{x^3+x^2+2x+5}{3x+\sqrt{x}} \quad \text{algebraic function}$$

e^x, $\ln x$, $\sin x$, $\arctan x$ — elementary transcendental functions

$\Phi(x)$, $\Gamma(x)$, $\zeta(s)$, $J_n(x)$, ...

special functions

To deal with $\Phi(x)$, either

(a) lengthy computations by hand)

b) computer / calculator

c) Table of values (p. 201)

<u>Standard</u> normal Random variable

= Normal RV with mean 0 and variance 1

PDF $\quad f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

CDF $\quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$