→ Properties of Expectation
→ Uniform Random Variables

Next HW
Problems: 5.12, 5.13, 5.15, 5.16, 5.18
5.23, 5.27
Theoretical: 5.2, 5.3, 5.9

If X has PDF $f(x)$

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$$

Theorem If $Y = g(X)$ (g some function)
then $E[Y] = E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx$

Corollary (1) $g(x) = x^n$

$$E[X^n] = \int_{-\infty}^{\infty} x^n f(x)\,dx$$

e.g. $\mathrm{Var}(X) = E[X^2] - (E[X])^2$

(2) $E[aX+b] = a\,E[X] + b$ $\quad a, b$ constant

Also true: $E[X+Y] = E[X] + E[Y]$

[Another definition of expectation

$E[X] = \sum_{\omega} X(\omega) P(\omega)$ $\quad$ sum over $\omega \in S$ outcomes

$E[X] = \int_S X\,dP$ $\quad$ integral w.r.t. probability measure]

Def Y is a non negative R.V. $(Y \geq 0)$

if $P(Y<0) = 0$

$$0 = P(Y<0) = \int_{-\infty}^{0} f(y)\,dy$$

implies $f(y) = 0$ (since $f(y) \geq 0$ always)
for $-\infty < y < 0$

Lemma Let Y be a non negative R.V.

$$E[Y] = \int_0^\infty P\{Y>y\}\,dy$$

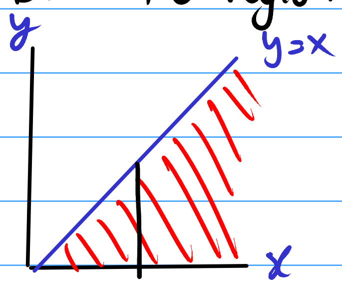
Remark $P\{Y>y\} = 1 - P\{Y \leq y\} = 1 - F(y)$
(CDF of Y)

Proof $$P\{Y>y\} = \int_y^\infty f(x)\,dx$$

$$\int_0^\infty P\{Y>y\}\,dy = \int_0^\infty \int_y^\infty f(x)\,dx\,dy$$

(iterated integral / double integral)
(read from inside out)
(x then y)

reverse order of integration (y then x)
Draw the region over which we are integrating



$$\int_0^\infty \int_0^x f(x)\,dy\,dx$$

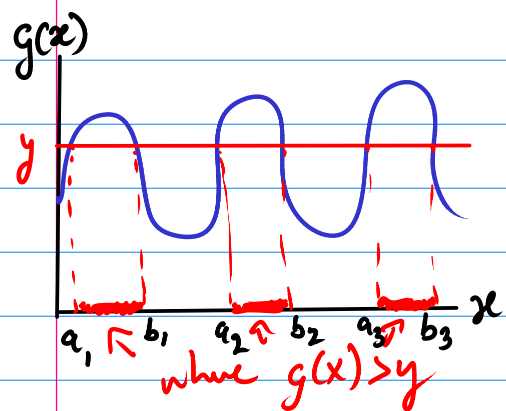
$$\int_0^\infty \int_0^x f(x)\,dy\,dx = \int_0^\infty \Big[y f(x) \Big]_0^x dx$$

$$= \int_0^\infty x f(x)\,dx = \int_0^\infty y f(y)\,dy = E[Y] \text{ by definition } \blacksquare$$

Theorem If g is a non negative function of x
Then $E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx$

Proof $E[g(X)] = \int_0^\infty P\{g(X) > y\}\,dy$ by Lemma

$$P\{g(X) > y\} = \int_{\substack{x \text{ such that} \\ g(x) > y}} f(x)\,dx$$



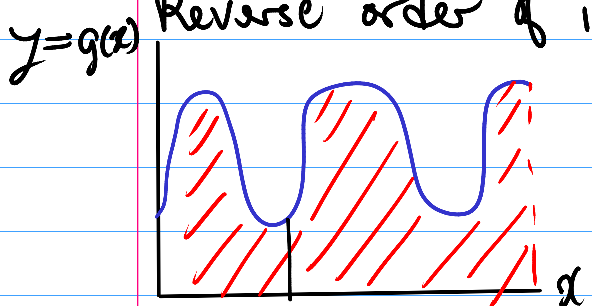
$$P\{g(X) > y\}$$

$$= \int_{a_1}^{b_1} f(x)dx + \int_{a_2}^{b_2} f(x)dx + \int_{a_3}^{b_3} f(x)dx$$

$$= \int_{\{x \mid g(x) > y\}} f(x)\,dx$$

$$E[g(X)] = \int_0^\infty \int_{\{x \mid g(x) > y\}} f(x)\,dx\,dy$$

Reverse order of integration



$$\int_{-\infty}^{\infty} \int_0^{g(x)} f(x)\,dy\,dx$$

$$\int_{-\infty}^{\infty}\int_{0}^{g(x)} f(x)\,dy\,dx = \int_{-\infty}^{\infty}\Big[y f(x)\Big]_{y=0}^{y=g(x)}\,dx$$

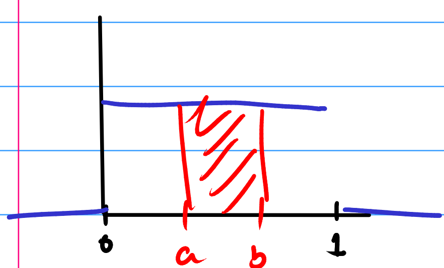
$$= \int_{-\infty}^{\infty} g(x) f(x)\,dx \qquad \text{what we wanted.}$$

$\blacksquare$

Uniform Random Variable
(continuous analog of "equally likely outcomes")

Def Uniform RV on $(0,1)=\{x \mid 0<x<1\}$ has PDF

$$f(x) = \begin{cases} 1 & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$



$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 1\,dx = 1$$

X = uniform on $(0,1)$

$$P(0<X<1) = 1$$

And if $0<a<b<1$

$$P(a\le X<b) = \int_a^b 1\,dx = b-a$$

only depends on length of the interval (a,b) not the position.

X is uniform on (α, β) $\alpha < \beta$

if X has PDF

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \alpha < x < \beta \\ 0 & \text{otherwise} \end{cases}$$