Continuous Random Variables: expectation and variance.

- Reminder test on Friday up to Lecture 22
- One 8.5" x 11" sheet of notes 2-sided
- Must be handwritten: no Xeroxes

Recall X is a continuous random variable

Possible values of X is some interval

$$(a,b) = \{x \mid a < x < b\}$$

$$[a,b] = \{x \mid a \leq x \leq b\}$$

or $[a,b) = \{x \mid a \leq x < b\}$

etc.

or $(-\infty, \infty)$ = all real numbers

X has a probability density function (PDF) $f(x)$ such that

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

Cumulative distribution function $F(a)$

$$F(a) = P(X \leq a) = P(-\infty \leq X \leq a)$$

$$= \int_{-\infty}^{a} f(x)\,dx$$

<u>Example</u> Amount of time a computer functions before breaking is a continuous random variable X

Let's say PDF of X is given by

$$f(x) = \begin{cases} \lambda e^{-x/100} & x \geq 0 \quad \text{in months} \\ 0 & x < 0 \end{cases}$$

λ is a normalization constant: choose it so that

$$P(-\infty \leq X \leq \infty) = \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} \lambda e^{-x/100}\,dx = \lambda \int_{0}^{\infty} e^{-x/100}\,dx$$

$$\left[\quad u = -\frac{x}{100} \quad du = -\frac{dx}{100} : \int_{0}^{a} e^{-x/100}\,dx = \int_{0}^{-\infty} e^{u}\,du\,(-100) \right.$$

$$= -100\int_0^{-\infty} e^u\,du = -100\left[e^u\right]_0^{-\infty} = -100\left(0 - e^0\right) = 100$$

$$\left[\int e^{au}\,du = \frac{1}{a}e^{au} + C\right]$$

$$1 = \lambda(100) \qquad \lambda = \frac{1}{100}$$

$$f(x) = \begin{cases} \frac{1}{100}e^{-x/100} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Q: $P(50 < X < 150) = \displaystyle\int_{50}^{150} \frac{1}{100}e^{-x/100}\,dx$

$$= \left[\frac{1}{100}(-100)e^{-x/100}\right]_{50}^{150} = \left[-e^{-x/100}\right]_{50}^{150}$$

$$= -e^{-150/100} - \left(-e^{-50/100}\right) = e^{-1/2} - e^{-3/2}$$

Expectation (Mean or Average Value)

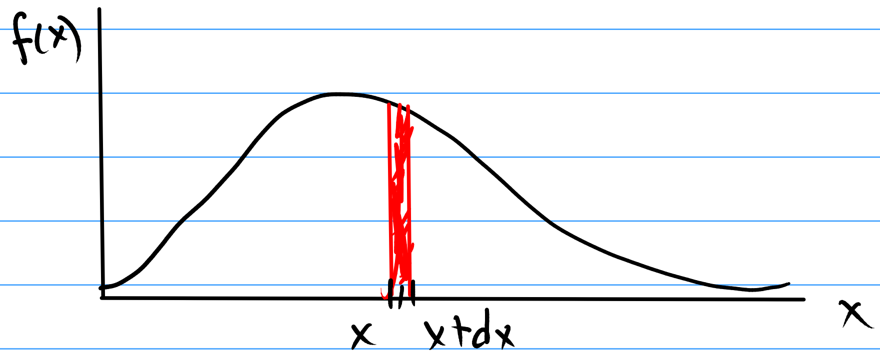
For discrete R.V. X has possible values x_i

$$P(a \le X \le b) = \sum_{a \le x_i \le b} P(X = x_i)$$

$$E[X] = \sum_{x_i} x_i \, P(X = x_i)$$

For continuous Random variable

$$f(x)\,dx \approx P(x \le X \le x + dx)$$



Break up real line into segments of width $\frac{1}{n}$

index the segments as $\left[\frac{i}{n}, \frac{i+1}{n}\right]$ $i = -\infty, \ldots, \infty$

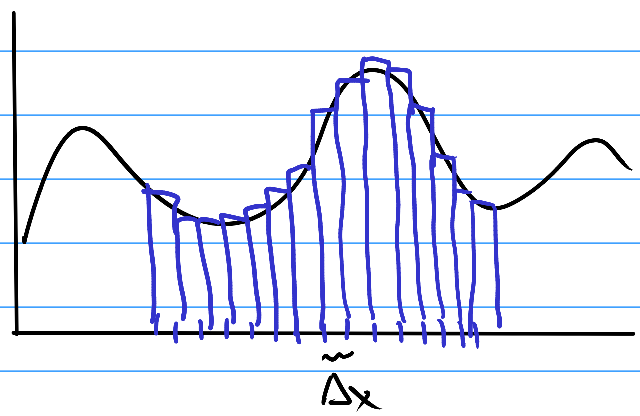
$$E[X] \approx \sum_{i=-\infty}^{\infty} \left(\frac{i}{n}\right) P\left(\frac{i}{n} \le X \le \frac{i}{n} + \frac{1}{n}\right)$$

$$\approx \sum_{i=-\infty}^{\infty} \left(\frac{i}{n}\right) f\left(\frac{i}{n}\right) \frac{1}{n}$$

We recognize this as a riemann sum

as $n \to \infty$, this sum converges to the integral

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$



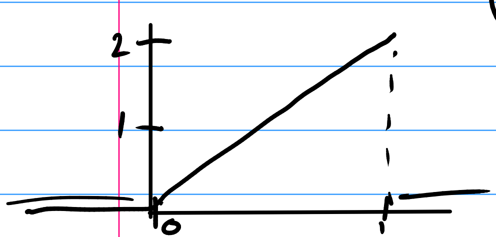
$\sum f(x_i)\, \Delta x_i$

Variance: definition interms of expectation is same a before

$$\mu = E[X] \text{ then } Var[X] = E\left[(X-\mu)^2\right]$$
$$= E[X^2] - \mu^2$$

Ex X has probability density function

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$



Can check this is normalised properly

Find $E[X] = \int_{-\infty}^{\infty} x f(x) dx = \int_0^1 x f(x) dx$

$$= \int_0^1 x(2x)\, dx = \int_0^1 2x^2 dx = \left[2\frac{x^3}{3}\right]_0^1$$

$$= \frac{2}{3} - 0 = \frac{2}{3}$$

$Var[X] = E[X^2] - \left(\frac{2}{3}\right)^2$

Fact (see next lecture for justification)

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_0^1 x^2 (2x) dx = \frac{1}{2}$$