More discrete probability distributions

NEXT Homework

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Theoretical 4.27, 4.29
Ch 5 Problems 5.1

Sources of random variables – Random Processes

Bernoulli trials $P(\text{success}) = p$ $P(\text{failure}) = 1-p$

$N(n)$ = # successes in n trials (binomial R.V. parameters (n, p))

Y = # of trials required to get first success (geometric Random variable)

$Z(r)$ = # of trials required to get first r successes (negative binomial random variable)

Binomial Random Variable X

- Parameters $n = \#$ of trials, $p =$ prob. of success
- Possible values $0, 1, 2, \ldots, n$
- Prob. Mass function $P(X=i) = \binom{n}{i} p^i (1-p)^{n-i}$
- $E[X] = np$
- $Var[X] = np(1-p)$
- SLOGAN: X represents the number of successes in n Bernoulli trials, where each trial has probability p of success
- EXAMPLE: A die is rolled 10 times
  let $X = \#$ times a 6 appears

then X is a binomial random variable with
$n = 10$, $p = 1/6$

- Remark: Binomial Random variable with $n=1$ is called Bernoulli random variable

Geometric random variable X

- Parameters p
- Possible values $1, 2, 3, \ldots, \infty$
- Probability mass function

$$P(X=i) = (1-p)^{i-1} p$$

EXAMPLE Roll a die until it comes up 6
$X = \#$ rolls required

$$P(X=3) = P(\text{not } 6, \text{not } 6, 6) = \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6}$$

$$P(X=i) = P(\text{not } 6\ (i-1) \text{ times}, 6) = \left(\frac{5}{6}\right)^{i-1}\left(\frac{1}{6}\right)$$

Remark on the name

$$\sum_{i=1}^{\infty} P(X=i) = \sum_{i=1}^{\infty} (1-p)^{i-1} p = p \sum_{i=1}^{\infty} (1-p)^{i-1}$$

$$\left(\sum_{i=1}^{\infty} a r^{i-1} = a \frac{1}{1-r} \right)$$

$$= p\,(1) \frac{1}{1-(1-p)} = p \cdot \frac{1}{p} = 1$$

- $E[X] = \frac{1}{p}$
- $Var[X] = \frac{1-p}{p^2}$

<u>Example</u> Sampling with replacement:

10 balls in an urn. One is white, 9 black
Balls are drawn, then replaced

$X = \#$ tries required to get the white ball

X is geometric random variable with parameter $p = \frac{1}{10}$

What is the probability that more than k tries are needed?

$$P(X \geq k) = \sum_{i=k}^{\infty} (1-p)^{i-1} p \qquad \text{first term} = (1-p)^{k-1} p$$

$$\text{ratio} = (1-p)$$

$$= (1-p)^{k-1} p \frac{1}{1-(1-p)} = (1-p)^{k-1}$$

Negative binomial random variable

- Parameters $r =$ # successes required, p
- Possible value $r, r+1, r+2, \ldots, \infty$
- Probability Mass function

$$P(X=i) = \binom{i-1}{r-1} p^r (1-p)^{i-r}$$

in order to have $X=i$, need exactly $r-1$ successes in $i-1$ trials, and need success on rth trial

$$\binom{i-1}{r-1} p^{r-1} (1-p)^{(i-1)-(r-1)} \cdot p$$

- Expectation $E[X] = \frac{r}{p}$
- Variance $Var[X] = \frac{r(1-p)}{p^2}$
- Geometric Random variable is negative binomial with $r=1$

Another process sampling without replacement

N balls in urn m are "red"
N−m are "blue"

Draw n balls without replacement

X = # of red balls in this sample of size n.

X is called Hypergeometric Random Variable

- Parameters n, m, N
- Possible values $0, 1, \dots, n$
- Probability mass function

$$P(X=i) = \frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}}$$

- Expectation $E[X] = n\frac{m}{N}$

(same as with replacement!)

- Variance $Var[X] = n\frac{m}{N}\left(1-\frac{m}{N}\right)\left(1-\frac{n-1}{N-1}\right)$

Take $m, N \to \infty$ in such a way that $p = \frac{m}{N}$ remains fixed, then the hyper-geometric random variable approaches a binomial random variable