Bernoulli, Poisson and Random Processes

(cf. pp. 152–154)

Recall: Poisson Random Variable X

parameter λ

Possible values $0, 1, 2, \ldots$ up to ∞

Prob. Mass function $p(X=i) = e^{-\lambda}\dfrac{\lambda^i}{i!}$

$$E[X] = \lambda$$

$Var[X] = \lambda$ $\quad SD[X] = \sqrt{\lambda}$ standard deviation

$$E[X] = \sum_{i=0}^{\infty} i\, e^{-\lambda}\frac{\lambda^i}{i!} = \sum_{i=1}^{\infty} e^{-\lambda}\frac{\lambda^i}{(i-1)!}$$

$$= \lambda \sum_{i=1}^{\infty} e^{-\lambda}\frac{\lambda^{i-1}}{(i-1)!} = \lambda \sum_{j=0}^{\infty} e^{-\lambda}\frac{\lambda^j}{j!} \quad (j=i-1)$$

$$= \lambda e^{-\lambda}\sum_{j=0}^{\infty}\frac{\lambda^j}{j!} = \lambda e^{-\lambda}e^{\lambda} = \lambda$$

$$E[X^2] = \sum_{i=0}^{\infty} i^2 e^{-\lambda}\frac{\lambda^i}{i!} = \lambda\sum_{i=1}^{\infty} i\, e^{-\lambda}\frac{\lambda^{i-1}}{(i-1)!}$$

$$= \lambda\sum_{j=0}^{\infty}(j+1)\, e^{-\lambda}\frac{\lambda^j}{j!}$$

$$= \lambda \Big[\underbrace{\sum_{j=0}^{\infty} j e^{-\lambda} \frac{\lambda^j}{j!}}_{\substack{\lambda \\ \text{by previous argument}}} + \underbrace{\sum_{j=0}^{\infty} e^{-\lambda} \frac{\lambda^j}{j!}}_{\substack{1 \text{ because sum of} \\ \text{all values of PMF}}} \Big]$$

$$= \lambda(\lambda + 1) = E[X^2]$$

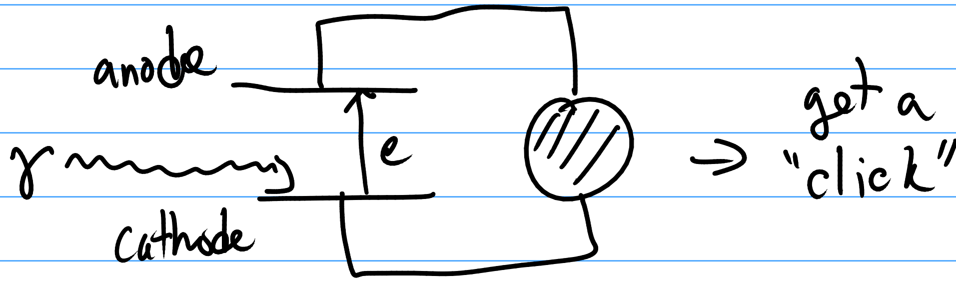
$$Var[X] = E[X^2] - (E[X])^2 = \lambda(\lambda+1) - \lambda^2$$

$$= \lambda$$

$$E[X^k] = \lambda E[(X+1)^{k-1}]$$

<u>Poisson Random Process</u>

<u>Example</u> Geiger Counter



Start counting clicks at time 0

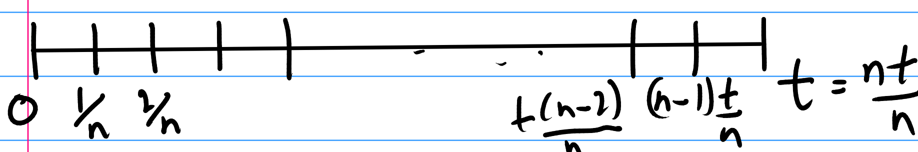
$N(t)$ = # of clicks in the time interval $[0, t)$
from 0 to t

For each t, $N(t)$ is a Random variable

But the whole family of Random variables depends on another parameter t, which can be any positive real number

So the collection of random variables $N(t)$ describes a continuous-time random process
(stochastic process)

Want to compute $P(N(t)=k)$



① Memorylessness: if two intervals of time are disjoint, then behavior of the process in those two intervals is independent.

② Some assumptions about the rate of "clicks"

$$P(\geq 1 \text{ click in interval of length } h) = \lambda h + o(h)$$

(depends only on the length of the interval)
roughly proportional to length of interval

λ = expected rate of clicks
units $1/\text{time}$

(Note not the same λ as before)

③ Can't get two clicks at exactly the same t

$$P(\geq 2 \text{ clicks in interval of length } h) = o(h)$$

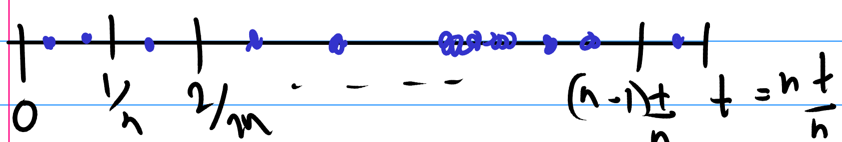
Landau Notation: $f(h) = o(h)$ as $h \to 0$

means $$\lim_{h \to 0} \frac{f(h)}{h} = 0$$

$f(h)$ goes to zero faster than h

so in the $h \to 0$ limit, $f(h)$ is negligible.

Subdivide the interval into bits of length $\frac{t}{n}$



(take $n \to \infty$)

$$P(N(t)=k) = P\left(\begin{array}{l}k \text{ intervals contain exactly 1 click} \\ n-k \text{ intervals contain exactly 0 clicks}\end{array}\right)$$

$$+ P\left(\begin{array}{l}N(t)=k \text{ and some subinterval} \\ \text{contains} \geq 2 \text{ clicks}\end{array}\right)$$

Step 1.

The second term is negligible

$$P(\text{some subinterval contains} \geq 2 \text{ clicks})$$

$$= P\left(\bigcup_{i=1}^{n} \{i\text{th subinterval contains} \geq 2 \text{ clicks}\right)$$

$$\leq \sum_{i=1}^{\infty} P(i\text{th subinterval contains} \geq 2 \text{ clicks})$$

(This is a corollary of inclusion-exclusion called Boole's inequality)

$$= \sum_{i=1}^{n} o\left(\frac{t}{n}\right) = n\,o\left(\frac{t}{n}\right) = t\,n\,o\left(\frac{1}{n}\right) \to 0 \text{ as } n \to \infty$$

$$\shortparallel$$

$$t\,\frac{o(1/n)}{1/n} \to 0 \text{ as } \frac{1}{n} \to 0$$

$P(N(t)=k) \approx P($ k subintervals contain 1 click, n−k contain 0 clicks $)$

approximate this probability by the binomial distribution

$$= \binom{n}{k}\left(\lambda\frac{t}{n} + o\left(\frac{t}{n}\right)\right)^{k}\left(1 - \frac{\lambda t}{n} - o\left(\frac{t}{n}\right)\right)^{n-k}$$

Like Bernoulli trials with $p = \lambda\frac{t}{n} + o\left(\frac{t}{n}\right)$

$$np = n\left(\lambda\frac{t}{n} + o\left(\frac{t}{n}\right)\right) = \lambda t + n\,o\left(\frac{t}{n}\right) \to \lambda t \text{ as } n\to\infty$$

$$\Rightarrow P(N(t)=k) = e^{-(\lambda t)}\frac{(\lambda t)^k}{k!}$$

Expected # of clicks in interval of length t is $E[N(t)] = \lambda t$

$$\text{Expected Rate} = E\left[\frac{N(t)}{t}\right] = \frac{1}{t}E[N(t)] = \lambda$$

Note: $o(h)$ notation will not be on the test!!!