Permutations with indistinguishable objects, and combinations

HOMEWORK has been posted

p. 16-17: 1, 3, 5, 8, 9, 11, 13, 28

p. 18-19: 2, 3, 4, 8, 10

Permutation = a way to order a collection of objects

Example: A math contest with 20 participants

Q. How many rankings are possible? (assume no ties)

A. $20! = 20 \cdot 19 \cdot 18 \cdot 17 \cdots \cdot 3 \cdot 2 \cdot 1$

Example 10 books

different subjects
4 math
3 Chemistry
2 history
1 Language

We want all books on a particular subject are grouped together how many orderings are possible?

$24 = 4! = $ # orderings of math books

$6 = 3! = $ # orderings of chem books

$2 = 2! = $ # " history

$1 = 1! = $ # " language

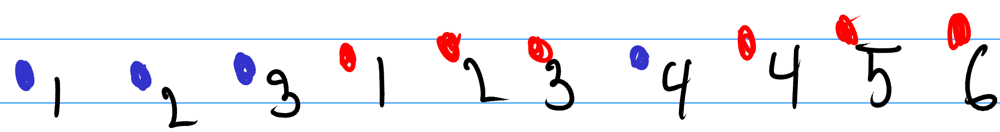
$4! = $ # orderings of subjects

$$\text{Total} = 4! \cdot 4! \cdot 3! \cdot 2! \cdot 1!$$

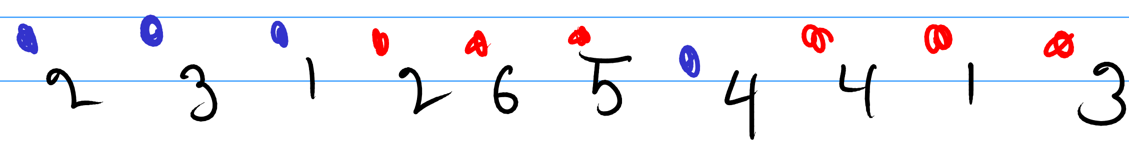
Permutations with some indistinguishable objects

Ex 10 beads, 4 blue, 6 red are lined up in a row.
How many possible arrangements?

One idea: first solve the problem, supposing the beads are distinguishable



How many arrangements with subscripts?
$10!$



Forget subscripts $\Rightarrow$ these two are the same.

How many times is this sequence of colors repeated?

permutations of blue subscripts = 4!

permutations of red subscripts = 6!

repeats = $4! \cdot 6!$

permutations with subscripts

= (# color sequences) · # repeats

$10! = (\#\text{ color sequences}) \cdot 4! \cdot 6!$

$$\#\text{ color sequences} = \frac{10!}{4!\,6!}$$

Example Anagrams of words with doubled letters

COMMITTEE How many anagrams?

$$\frac{9!}{2! \cdot 2! \cdot 2!}$$

n objects of r types

n_1 of type 1 all alike

n_2 of type 2 "

⋮ ⋮

n_r of type r

$$\# \text{ arrangements} = \frac{n!}{n_1!\, n_2! \cdots n_r!}$$

Combinations

We have a collection of n objects, and we want to choose r of them

In how many ways can this choice be made?

Now, the objects are distinguishable, But we don't care about the order in which the objects are chosen.

"n choose 1" $= n$

"n choose 2" $= n \cdot (n-1)/2$

$\{1, 2, 3, 4, \dots, n\}$

But $(1,2)$ and $(2,1)$ are the same collection of elements of this set

"n choose r" $= \dfrac{(\text{\# ordered subsets of size } r)}{(\text{\# permutations of a set of size } r)}$

$(\text{\# ordered subsets of size } r)$

$$= \underset{\substack{\uparrow \\ \text{1st}}}{n} \cdot \underset{\substack{\uparrow \\ \text{2nd}}}{(n-1)} \cdot \underset{\substack{\uparrow \\ \text{3rd}}}{(n-2)} \cdots \underset{\substack{\uparrow \\ r\text{th}}}{\overset{(n-(r-1))}{\overbrace{(n-r+1)}}}$$

$$= n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) \cdot \frac{(n-r) \cdots (1)}{(n-r) \cdots (1)}$$

$$= \frac{n!}{(n-r)!}$$

$\#$ permutations of set of size r $= r!$

"n choose r" $= \frac{n!}{(n-r)!} \cdot \frac{1}{r!}$

$$= \frac{n!}{(n-r)!\,r!} =: \binom{n}{r}$$ Binomial coefficient

Ex How many 5-card poker hands are possible?

"52 choose 5" $= \binom{52}{5} = \frac{52!}{47!\,5!}$

Ex 12 people divided into 3 committees of sizes 3, 4, and 5

How many ways?

$$\underbrace{\binom{12}{3}}_{\text{1st comm.}} \cdot \binom{9}{4} \cdot \binom{5}{5}$$

$$\binom{5}{5} = \frac{5!}{0!\,5!} = 1$$

NOTE: $0! = 1$

$1! = 1$

$n \cdot (n-1)! = n!$

Prove $\binom{n}{r} = \binom{n}{n-r}$

Proof 1 use formula

$$\binom{n}{n-r} = \frac{n!}{\underbrace{(n-(n-r))}_{r}!\,(n-r)!}$$

$$= \frac{n!}{r!\,(n-r)!} = \binom{n}{r}$$

Proof 2 combinatorial:

$\binom{n}{r}$ = # subsets of size r

$\binom{n}{n-r}$ = # subsets of size $n-r$

Given subset of size r, look at the complement (what's leftover). This is a set of size $n-r$.