Binomial Cont'd / Poisson R.V.s

Recall: Binomial Random variable

X takes values $0, 1, 2, \ldots, n$

$$P(X=i) = \binom{n}{i} p^i (1-p)^{n-i}$$

This represents the # of successes in n independent Bernoulli trials with p = probability of success

What is expectation and variance ?

Involves recursive relationship between the moments of X

Trick: If X is a binomial RV w/ parameters (n,p) and Y is a binomial RV w/ parameter $(n-1, p)$

Moments $E[X^k] = np\, E[(Y+1)^{k-1}]$ **

$E[X] = np\, E[(Y+1)^0] = np \cdot 1 = np$

$E[X^2] = np\, E[(Y+1)^1] = np(E[Y]+1)$

$= np((n-1)p+1) = n^2p^2 + np(1-p)$

$$Var[X] = E[X^2] - (E[X])^2$$
$$= n^2p^2 + np(1-p) - (np)^2$$
$$= np(1-p)$$

So: $E[X] = np \Rightarrow E\left[\frac{X}{n}\right] = p$

$\frac{X}{n}$ is average number of successes in n trials

Expected proportion of successes is p.

Shape binomial probability mass function with parameters (n, p)

$p(k) = P(X=k)$, then $p(k)$ first increases monotonically, then decreases monotonically

it obtains its maximum value when

$k = \lfloor (n+1)p \rfloor$ ← rounded down

Poisson Random Variable X with parameter λ

X has possible values $0, 1, 2, 3, \ldots$ up to ∞

$$p(i) = P(X=i) = e^{-\lambda} \frac{\lambda^i}{i!}$$

Recall $$\sum_{i=0}^{\infty} p(i) = e^{-\lambda} \Big(\underbrace{\sum_{i=0}^{\infty} \frac{\lambda^i}{i!}}_{\text{taylor series for } e^{\lambda}} \Big) = e^{-\lambda} \left(e^{\lambda} \right) = 1$$

Poisson random variable also known as "law of rare events"

E.g. # misprints per page in a book

people who live to 100

wrong telephone numbers dialed in day

α-decays per unit time per unit mass in a radioactive material.

Poisson distribution approximates binomial distribution when n is large and p is small and np is of moderate size. $(\lambda = np)$

X is binomial w/ parameters (n, p)

set $\lambda = np$ hence $p = \dfrac{\lambda}{n}$

$$P(X=i) = \binom{n}{i} p^i (1-p)^{n-i} = \frac{n!}{(n-i)!\, i!} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i}$$

$$= \frac{(n)(n-1)(n-2)\cdots(n-i+1)}{n^i} \frac{\lambda^i}{i!} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^i}$$

Take $n \to \infty$ $\dfrac{n(n-1)\cdots(n-i+1)}{n^i} \approx \dfrac{n^i}{n^i} = 1$

$$\left(1 - \frac{\lambda}{n}\right)^i \to 1^i = 1$$

$\left(1 - \dfrac{\lambda}{n}\right)^n \to e^{-\lambda}$ because $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n = e$

$$P(X=i) \approx e^{-\lambda} \frac{\lambda^i}{i!}$$

<u>Examples</u> Suppose # typos per page in a book is a Poisson RV. with parameter $\lambda = \frac{1}{2}$

$$P(X \geq 1) = 1 - P(X=0) = 1 - e^{-\lambda} = 1 - e^{-1/2} = .393$$

$$E[X] = \sum_{i=0}^{\infty} i\, e^{-\lambda} \frac{\lambda^i}{i!} = \sum_{i=1}^{\infty} \frac{e^{-\lambda}\lambda^i}{(i-1)!}$$

$$= \lambda e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!} = \lambda e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}$$

(let $j = i-1$)

$$= \lambda e^{-\lambda} e^{\lambda} = \lambda$$