Variance / Binomial Random Variables

Next HW:

Problems: 4.30, 4.33, 4.35, 4.38, 4.41, 4.57, 4.59, 4.61

Theoretical Exercises: 4.16, 4.19

Reference for previous lecture §4.4 §4.9

$$E[X] = \sum_{\theta} X(\theta)\, P\{\theta\}$$

$$\mathrm{Var}[X] = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

Ex X = fair die roll $X = 1, 2, 3, 4, 5, 6$

$$E[X] = 3.5$$

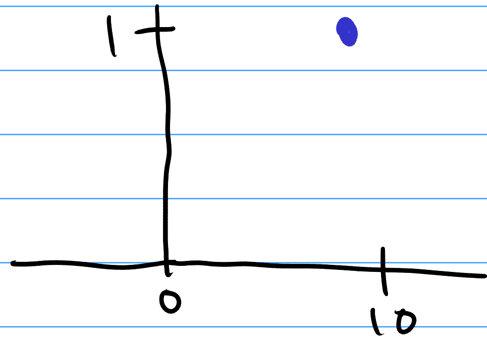
$\mathrm{Var}[X]$? $E[X^2] = 1^2\left(\frac{1}{6}\right) + 2^2\left(\frac{1}{6}\right) + 3^2\left(\frac{1}{6}\right) + 4^2\left(\frac{1}{6}\right) + 5^2\left(\frac{1}{6}\right) + 6^2\left(\frac{1}{6}\right)$

$= 91\left(\frac{1}{6}\right)$

$$\mathrm{Var}[X] = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12} \approx 2.9$$

Variance measures the "width" of the range over which X is distributed

$W = 10$ with prob 1



$E[W] = 10$

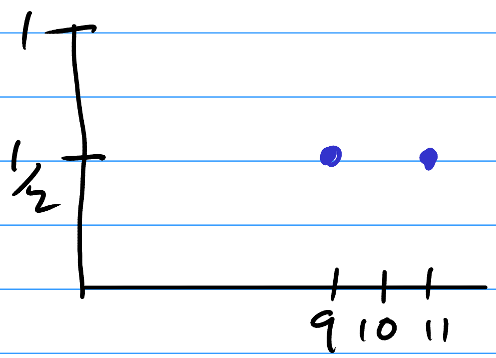
$Var[W] = E[(W-10)^2]$

$(W-10)^2 = 0$ with prob 1

$Var[W] = 0$

$Y = 11$ with Prob. $\frac{1}{2}$
$Y = 9$ with prob $\frac{1}{2}$

$E[Y] = 10$



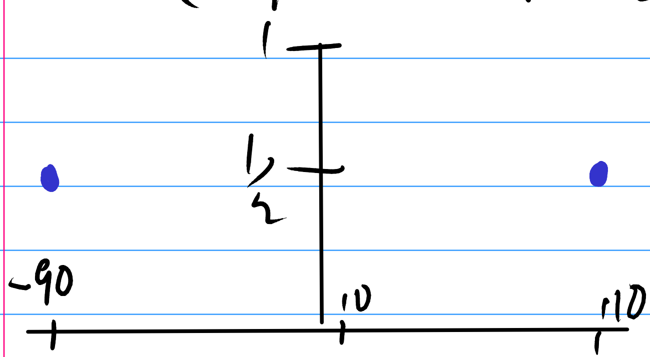
$Var[Y] = E[(Y-10)^2]$

$$(Y-10) = \begin{cases} 1 & \text{prob } 1/2 \\ -1 & \text{prob } 1/2 \end{cases}$$

$Var[Y] = (1)^2 \frac{1}{2} + (-1)^2 \frac{1}{2} = 1$

$$Z = \begin{cases} 110 & \text{with prob. } 1/2 \\ -90 & \text{with prob } 1/2 \end{cases}$$

$E[Z] = 110 \frac{1}{2} + (-90)\frac{1}{2} = 10$



$(Z-10)^2 = (100)^2 = 10000$ with prob 1

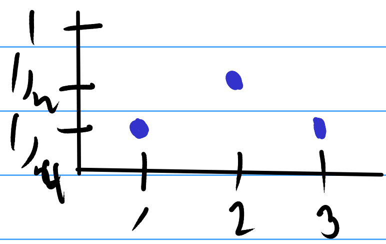
$Var[Z] = 10000$

Standard Deviation $SD[X] = \sqrt{Var[X]}$

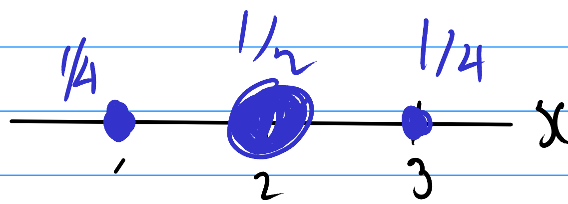
$SD[W] = 0$ $\quad SD[Y] = 1$ $\quad SD[Z] = 100$

So variance measures how widely dispersed the values of X are.

Analogy: If think of the probability mass function as describing the masses of certain bodies arranged on a line:



$\longleftrightarrow$



$E[X] \Leftrightarrow$ center of mass

$Var[X] \Leftrightarrow$ moment of inertia about the center of mass

Bernoulli trials and Binomial Random Variables

Recall Bernoulli trial $P(\text{success}) = p$

$P(\text{failure}) = 1-p$

Suppose X is a random variable which is 1 on success and 0 on failure

Prob. mass function $p(0) = 1-p$

$p(1) = p$

X is "Bernoulli Random variable with parameter p"

Now suppose we do n independent bernoulli trials

X = # of successes obtained in n trials

X takes values $0, 1, \ldots, n$

$$p(i) = P(X=i) = \binom{n}{i} p^i (1-p)^{n-i}$$

X is called "Binomial random variable with parameters (n, p)"

# of trials

prob of success per trial

$$1 \overset{?}{=} \sum_{i=0}^{n} p(i) = \sum_{i=0}^{n} \binom{n}{i} \underset{\underset{x}{\shortparallel}}{p^i} \underset{\underset{y}{\shortparallel}}{(1-p)}^{n-i} = (x+y)^n = (p+1-p)^n$$

$$= 1^n = 1$$

Examples

Basic (seen before) Five fair coins are flipped

$X = \#$ heads

Then X is a binomial Random variable w/ parameters $(n = 5, p = \frac{1}{2})$

$$P(X = 2) = \binom{5}{2} \left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)^3 = \frac{10}{32}$$

Example Sell cases of wine. each case contains 20 bottles

Prob of bad bottle $= .05$

Money-back gaurantee that case will contain no more than 1 bad bottle.

$X = \#$ of bad bottles in a case (20 bottles)

$P(\text{have to give money back}) = P(X \geq 2)$

$= 1 - P(X=0) - P(X=1)$

X is binomial R.V. with parameters

$(n = 20,\ p = .05)$

Bernoulli trial = check whether a bottle is bad

P(success) = P(bottle is bad) = .05

P(failure) = P(bottle is good) = .95

$P(X \geq 2) = 1 - P(X=0) - P(X=1)$

$= 1 - \binom{20}{0}(.05)^0(.95)^{20} - \binom{20}{1}(.05)^1(.95)^{19}$

$= .26$