PROPERTIES OF EXPECTATION and VARIANCE

Recall Random variable X

Probability mass function $p(x) = P(X=x)$

Expecation $E[X] = \sum_x x\, p(x)$

where x ranges through all possible values of X.

Next concept:

Def If X is a random variable with $\mu = E[X]$

(μ = expectation or mean of X)

The variance is

$$Var[X] = E\left[(X-\mu)^2\right] = E\left[(X-E[X])^2\right]$$

$$Var[X] = E[Y] \qquad Y = (X-\mu)^2$$

or in other words if $g(t) = (t-\mu)^2$

then $Y = g(X)$

Ex X a random variable taking values $-1, 0, 1$

$P\{X=-1\} = .2$

$P\{X=0\} = .5$

$P\{X=1\} = .3$

$E[X] = (-1)(.2) + 0(.5) + 1(.3)$
$= .1$

$Var(X)$: $Y = (X-.1)^2$

possible values of $X-.1 = \begin{cases} .9 \\ -.1 \\ -1.1 \end{cases}$

possible values of $Y = \begin{cases} (.9)^2 = .81 \\ (-.1)^2 = .01 \\ (-1.1)^2 = 1.21 \end{cases}$

$P(Y=.81) = .3$
$P(Y=.01) = .5$
$P(Y=1.21) = .2$

$Var[X] = E[Y]$
$= (.81)(.3) + (.01)(.5) + 1.21(.2)$

$Z = X^2$ possible values of Z $0, 1$

$P(Z=0) = P(X=0) = .5$

$P(Z=1) = P(X=1) + P(X=-1) = .3 + .2 = .5$

$E[Z] = 0(.5) + 1(.5) = .5$

What about variable X takes values $0, 1, 2, 3, 4, \ldots$

$p(i)$ probability mass function of X

$$Y = \sin\left(\frac{2\pi}{13} X\right)$$

PMF of Y? $\quad p(y) = P(Y = y) = P\left(\sin\left(\frac{2\pi}{13}X\right) = y\right)$

To find expectation of Y it is not necessary to find the probability mass function of Y. In fact you only need PMF of X.

ANOTHER (more fundamental) DEFINITION OF EXPECTATION

Assume S = sample space is discrete

each outcome σ in S has some prob $P\{\sigma\}$

$$P(E) = \sum_{\sigma \text{ in } E} P\{\sigma\}$$

Think of a random variable as a function from S to the set of real numbers

$X : S \to \mathbb{R} \qquad X(\sigma) =$ value of X on outcome σ

$$E[X] = \sum_{\omega} X(\omega)\, P\{\omega\}$$ sum over all outcomes.

Why is it the same as "PMF definition"?

Group terms according to the value of X

$$\sum_{\omega} X(\omega) P\{\omega\} = \sum_{x} \sum_{\omega \text{ such that } X(\omega)=x} X(\omega) P\{\omega\}$$

$$= \sum_{x} \sum_{\omega \text{ such that } X(\omega)=x} x\, P\{\omega\} = \sum_{x} x \sum_{\substack{\omega \\ \text{s.t.} \\ X(\omega)=x}} P\{\omega\}$$

$$= \sum_{x} x\, P(X=x) = \sum_{x} x\, p(x)$$

<u>Proposition</u> If Random variable Y is a function of X

$Y = g(X)$ and $p(x)$ is PMF of X.

$$E[Y] = \sum_{\substack{x \\ \text{possible} \\ \text{value} \\ \text{of } X}} g(x)\, p(x)$$

$\boxed{\text{Don't need PMF of } Y}$

Proof $E[Y] = \sum_{\sigma} Y(\sigma)\, P\{\sigma\}$

$$= \sum_{\sigma} g(X(\sigma))\, P(\sigma)$$

$$= \sum_{x} \sum_{\substack{\sigma \text{ s.t.} \\ X(\sigma)=x}} g(x)\, P(\sigma)$$ group terms according to value of X

$$= \sum_{x} g(x) \sum_{\substack{\sigma \text{ s.t.} \\ X(\sigma)=x}} P(\sigma) = \sum_{x} g(x)\, P(X=x)$$

$$= \sum_{x} g(x)\, p(x)$$

$P(X=-1) = .2$ $\quad Y = X^2$
$P(X=0) = .5$
$P(X=1) = .3$ $\quad E[Y] = (-1)^2(.2) + 0^2(.5) + (1)^2(.3)$
$= .5$

COMMON APPLICATIONS

$Y = X^n \quad E[X^n] = \sum_{x} x^n\, p(x)$

$E[X^n]$ is called the n th moment of X

$$E[aX+b] = \sum_x (ax+b)\,p(x)$$

$$= a\sum_x x p(x) + b\sum_x p(x)$$

$$= a\,E[X] + b\,(1)$$

$$= aE[X] + b$$

IMPORTANT PROPERTY in Section 4.9 ~~NOT IN CHAP 4~~

if X and Y are any R.V.s on the same sample space then

$$E[X+Y] = E[X] + E[Y]$$

$$E[X+Y] = \sum_\sigma \big(X(\sigma)+Y(\sigma)\big)\,P(\sigma)$$

$$= \sum_\sigma X(\sigma)P(\sigma) + \sum_\sigma Y(\sigma)P(\sigma)$$

$$= E[X] + E[Y]$$

$E[X] = \mu$

$$Var[X] = E[(X-\mu)^2]$$

ANOTHER FORMULA (often more convenient)

$$Var[X] = E[X^2] - \mu^2 = E[X^2] - (E[X])^2$$

$$(X-\mu)^2 = X^2 - 2\mu X + \mu^2$$

$$E[X^2 - 2\mu X + \mu^2] = E[X^2] - 2\mu \underbrace{E[X]}_{\mu} + \mu^2$$

$$= E[X^2] - \mu^2$$