EXPECTED VALUE OR EXPECTATION

So far Random Variables X

Probability mass function $p(x) = P(X = x)$

Cumulative distribution function $F(x) = P(X \leq x)$

If X is a discrete Random variable with probability mass function $p(x)$

then the expectation of X is

$$\underbrace{E[X]} = \sum_{x:p(x)>0} x\,p(x) = \sum_{x \text{ a possible values}} x\,P(X=x)$$

use brackets because it depends on all possible values of X

Ex 1 $X = 0$ or 1 each with prob. $1/2$

$$p(0) = \frac{1}{2} = p(1)$$

$$E[X] = 0\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right) = \frac{1}{2}$$

Ex 2 $p(0) = \frac{1}{3}$ $p(1) = \frac{2}{3}$ $E[X] = 0\frac{1}{3} + 1\frac{2}{3} = \frac{2}{3}$

$E[X]$ is an average or mean value of X

more precisely, a <u>weighted average</u> where each possible value is weighted by the probability that it will occur.

<u>Ex</u> X = roll of a fair 6-sided die

$$x = 1, 2, 3, 4, 5, 6$$

$$E[X] = \sum_{i=1}^{6} i\left(\frac{1}{6}\right) = \frac{1}{6}\sum_{i=1}^{6} i = \frac{1}{6}21 = \frac{7}{2} = 3.5$$

<u>Ex</u> Here's a game

$$P(\text{win}) = \frac{99}{100} \qquad P(\text{lose}) = \frac{1}{100}$$

$$\left.\begin{array}{l} \text{win} \Rightarrow \text{gain } \$1 \\ \text{lose} \Rightarrow \text{lose } \$1000000 \end{array}\right\} \text{payout}$$

X = amount we win or lose

$$E[X] = 1\left(\frac{99}{100}\right) + (-1000000)\frac{1}{100} \approx -10000$$

THEORETICAL EXAMPLE

Let A be an event

I is an <u>indicator</u> variable:

$I = 1$ if A occurs (i.e. for any outcome in A)

$I = 0$ if A^c occurs

$E[I] = ?$

possible values: 0, 1

PMF: $p(0) = P(I=0) = P(A^c) = 1 - P(A)$
$p(1) = P(I=1) = P(A)$

$$E[I] = 0\,p(0) + 1\,p(1) = p(1) = P(A)$$

Reprove inclusion-exclusion:

E, F events $\quad P(E \cup F) = P(E) + P(F) - P(EF)$

I_E indicator variable for E

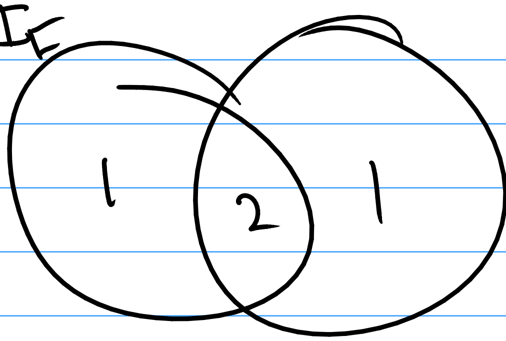
I_F " " " F

$$I_E + I_F = \begin{cases} 0 & \text{if neither } E \text{ nor } F \text{ occurs} \\ 1 & \text{if exact one of } E \text{ or } F \text{ occurs} \\ 2 & \text{if both occur.} \end{cases}$$

$$I_{EF} = \begin{cases} 1 & \text{if both } E, F \text{ occur} \\ 0 & \text{otherwise} \end{cases}$$

$$I_{E \cup F} = \begin{cases} 1 & \text{if } E \text{ or } F \text{ or both} \\ 0 & \text{otherwise.} \end{cases}$$

$I_E + I_F$



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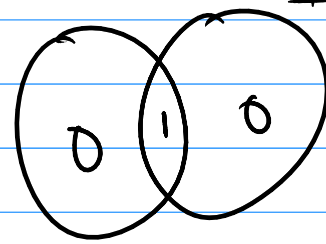
I_{EF}



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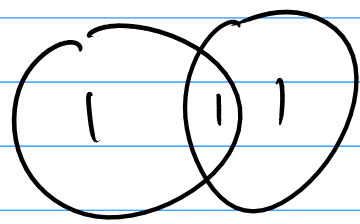


$= I_{E \cup F}$

$$I_{E \cup F} = I_E + I_F - I_{EF}$$

$$E[I_{E \cup F}] = E[I_E] + E[I_F] - E[I_{EF}]$$

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

100 ball in urn
20 blue
30 green
50 red

Ball is drawn at random and we win amount equal to the number of balls of that color.

X = amount we win

$E[X]$? Possible values: $X = 20, 30, 50$

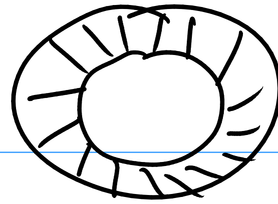
PMF
$$p(20) = 20/100 = .2$$
$$p(30) = 30/100 = .3$$
$$p(50) = 50/100 = .5$$

$$E[X] = 20(.2) + 30(.3) + 50(.5)$$
$$= 4 + 9 + 25 = 38$$

Urn contains just one ball of each color
payouts the same Y = winnings

$$E[Y] = 20\left(\tfrac{1}{3}\right) + 30\left(\tfrac{1}{3}\right) + 50\left(\tfrac{1}{3}\right)$$
$$= \frac{100}{3} = 33\tfrac{1}{3}$$

Roulette



Version 1 wheel w/ 18 red places, 18 black places] # 1-36

X Bet on color red/black → payout 1 to 1 ratio

Y Bet on particular # → payout 35 to 1 ratio

X: lose → $X=-1$ $P(\text{lose})=\frac{1}{2}$

win → $X=1$ $P(\text{win})=\frac{1}{2}$

$$E[X]=(-1)\left(\frac{1}{2}\right)+(1)\left(\frac{1}{2}\right)=0$$

This game is "fair"

Y: lose → $Y=-1$ $P(\text{lose})=35/36$

win → $Y=35$ $P(\text{win})=1/36$

$$E[Y]=(-1)\left(\frac{35}{36}\right)+(35)\left(\frac{1}{36}\right)=0$$

Also <u>fair</u>

Change the wheel 18 red, 18 black } # 1–36

two green places 0, 00

Total of 38 places.

payouts are the same as before

X bet on red $X = -1, 1$

$$E[X] = (-1)\left(\frac{20}{38}\right) + 1\left(\frac{18}{38}\right) = -\frac{2}{38} = -\frac{1}{19}$$

Y = bet on #5 $Y = -1, 35$

$$E[Y] = (-1)\left(\frac{37}{38}\right) + (35)\left(\frac{1}{38}\right) = -\frac{2}{38} = -\frac{1}{19}$$

So lose about $\$\frac{1}{19} = 5.3¢$ on the dollar, per game