$P(\cdot \mid F)$ is a probability measure

Exam on Friday. Wednesday will be review.

Office hours M 1-2, 3-4, W 9:30-10:30

Let F be some fixed event

define $Q(E) = P(E \mid F)$

$Q(\cdot) = P(\cdot \mid F)$ satisfies all properties of a probability measure

<u>Prop</u> $0 \le P(E \mid F) \le 1$ $\qquad 0 \le Q(E) \le 1$

$P(S \mid F) = 1$ $\qquad Q(S) = 1$

if $E_1, E_2, \ldots$ are mutually exclusive

$$P\left(\bigcup_{i=1}^{\infty} E_i \,\Big|\, F\right) = \sum_{i=1}^{\infty} P(E_i \mid F)$$

$$Q\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} Q(E_i)$$

Anything that follows from axioms, has a "conditional" version

$$P(E^c \mid F) = 1 - P(E \mid F)$$

[Assume $P(F) \neq 0$]

$$P(E_1 \cup E_2 | F) = P(E_1|F) + P(E_2|F) - P(E_1E_2|F)$$

Proof of Prop

$$0 \leq P(E|F) \leq 1$$

$$P(E|F) = \frac{P(EF)}{P(F)} \geq 0$$

$$P(EF) \leq P(F) \quad b/c \quad EF \subset F$$

$$\frac{P(EF)}{P(F)} \leq 1$$

$$P(S|F) = \frac{P(SF)}{P(F)} = \frac{P(F)}{P(F)} = 1$$

$$P\left(\bigcup_{i=1}^{\infty} E_i \Big| F\right) = \frac{P\left(\left(\bigcup_{i=1}^{\infty} E_i\right) F\right)}{P(F)}$$

$$\left(\bigcup_{i=1}^{\infty} E_i\right) F = \bigcup_{i=1}^{\infty} (E_i F) \quad \text{distributive law}$$

$$E_i F \, E_j F \subseteq E_i E_j = \emptyset \quad b/c \ E_i \text{ mut. exc.}$$

$$\frac{P\left(\bigcup_{i=1}^{\infty} E_i F\right)}{P(F)} = \sum_{i=1}^{\infty} \frac{P(E_i F)}{P(F)} = \sum_{i=1}^{\infty} P(E_i|F)$$

Independent events E_1, E_2 $P(E_1E_2) = P(E_1)P(E_2)$

Conditionally independent events

$$P(E_1E_2 \mid F) = P(E_1 \mid F)\,P(E_2 \mid F)$$

Examples Insurance company

A = accident prone
A_1 = has accident in 1st year
A_2 = has accident in 2nd year

$P(A) = .3$ $P(A_i \mid A) = .4$
$P(A_i \mid A^c) = .2$

Assume A_1, A_2 conditionally independent

Previously we saw $P(A_i) = .26$

New Q: $P(A_2 \mid A_1)$ Accident in 2nd year given accident in first year.

Condition on A, whether person is accident.
Also, take all probabilities conditional on A_1

$$P(A_2 \mid A_1) = P(A_2 \mid AA_1)P(A \mid A_1) + P(A_2 \mid A^cA_1)P(A^c \mid A_1)$$

$$P(A_2 \mid AA_1) \underset{\uparrow}{=} P(A_2 \mid A) = .4$$

because different years independent

$$P(A_2 \mid A^c A_1) = P(A_2 \mid A^c) = .2$$

$$P(A \mid A_1) = \frac{P(A_1 \mid A)P(A)}{P(A_1)} = \frac{(.4)(3)}{.26} = \frac{6}{13}$$

$$P(A^c \mid A_1) = 1 - P(A \mid A_1) = \frac{7}{13}$$

$$P(A_2 \mid A_1) = (.4)\frac{6}{13} + .2\frac{7}{13} \approx .29$$

Laplace's rule of succession:

$k+1$ coins in a box ith coin comes up heads with probability i/k. $(i = 0, \ldots, k)$

We flip coin n tries, and get n Heads what is prob. that $(n+1)$st flip is heads?

C_i = ith coin is selected. $P(C_i) = \frac{1}{k+1}$

F_n = first n flips are heads.

H = $(n+1)$st flip is heads.

$P(F_n \mid C_i) = (i/k)^n$ (Bernoulli trials with $p = i/k$)

$P(H \mid C_i) = i/k$

The different flips are conditionally independent, given C_i

$$P(H|F_n) = \sum_{i=0}^{k} P(H|F_n C_i)\, P(C_i|F_n)$$

$$P(H|F_n C_i) = P(H|C_i) = i/k$$

because H and F_n are independent given C_i

Bayes Formula

$$P(C_i|F_n) = \frac{P(F_n|C_i)\,P(C_i)}{P(F_n)}$$

$$= \frac{P(F_n|C_i)P(C_i)}{\sum_{j=0}^{k} P(F_n|C_j)\,P(C_j)} = \frac{(i/k)^n \frac{1}{k+1}}{\sum_{j=0}^{k} (j/k)^n \frac{1}{k+1}}$$

$$= \frac{(i/k)^n}{\sum_{j=0}^{k} (j/k)^n}$$

$$P(H|F_n) = \sum_{i=0}^{k} (i/k) \frac{(i/k)^n}{\sum_{j=0}^{k} (j/k)^n} = \frac{\sum_{i=0}^{k} (i/k)^{n+1}}{\sum_{j=0}^{k} (j/k)^n}$$

if k is large

$$P(H|F_n) \approx \frac{n+1}{n+2}$$