Independent Events

Events are independent if the occurance of one event doesn't affect the probability of the other event.

In general $P(E|F) = \dfrac{P(EF)}{P(F)} \neq P(E)$

But if equality holds

$P(E|F) = P(E)$ or equivalently $P(EF) = P(E)P(F)$

Then we say E and F are independent.

Examples 1 Multiple trials of given experiment

The outcomes of different trials are independent

Roll a die n times (n trials)

$E = \{\text{even \# on first roll}\}$ $F = \{\text{get odd \# on 10th roll}\}$

Example 2 Roll 2 dices

E_1 = sum is 6
E_2 = sum is 7
F = first die is 4

E_1 and E_2 are not independent (in fact mut. exclusive)

(1,5) (2,4) (3,3) (4,2) (5,1)

E_1 and F $\quad P(E_1) = \frac{5}{36}$

$P(F) = \frac{1}{6}$ $\quad P(E_1F) = \frac{1}{36}$

$\frac{5}{36} \cdot \frac{1}{6} \overset{?}{\neq} \frac{1}{36}$ NO not indep

E_2 and F $\quad P(E_2) = \frac{6}{36} = \frac{1}{6}$

(1,6), (2,5) (3,4) (4,3) (5,2) (6,1)

$P(E_2F) = \frac{1}{36}$

$\frac{1}{6} \cdot \frac{1}{6} \overset{?}{=} \frac{1}{36}$ Yes these are independent.

Prop If E and F are independent so are E and F^c

also E^c and F

also E^c and F^c

$E = EF \cup EF^c$ $\quad P(E) = P(EF) + P(EF^c)$

mut exclusive

$P(E) = P(E)P(F) + P(EF^c)$

$[1-P(F)]\,P(E) = P(EF^c)$

$P(F^c)\,P(E) = P(EF^c)$ $\square$

When are E, F, and G independent?

Roll 2 dice $\quad E$ = sum is 7
F = first die is 4
G = second die is 3.

$$\begin{tikzcd} E \arrow[rr, leftrightarrow, "\text{indep}"] \arrow[dr, leftrightarrow, swap, "\text{indep}"] & & F \arrow[dl, leftrightarrow, "\text{indep.}"] \\ & G & \end{tikzcd}$$

$$P(E|FG) = 1$$

$$\frac{P(EFG)}{P(FG)} = 1$$

$$P(EFG) = P(FG)$$
$$\neq P(E)P(FG)$$

Definition
3 events E F and G are independent (as a set of 3)

If $\quad P(EFG) = P(E)\,P(F)\,P(G)$

and $\quad P(EF) = P(E)\,P(F)$
$\quad P(EG) = P(E)\,P(G)$
$\quad P(FG) = P(F)\,P(G)$

n events $E_1, E_2, \ldots, E_n$ are independent

if for any subcollection of the events

$$E_{i_1}, E_{i_2}, \ldots, E_{i_r}$$

we have

$$P(E_{i_1} E_{i_2} \cdots E_{i_r}) = P(E_{i_1})\,P(E_{i_2}) \cdots P(E_{i_r})$$

i.e., the product rule holds for any subcollection of the events.

E.g. of E, F, G independent

E independent from FG, $F \cup G$

NB. Mutually exclusive $EF = \emptyset \Rightarrow E$ and F are not independent

If $E \subset F \Rightarrow E$ and F are not independent

Main example Bernoulli Trials

A Bernoulli trial is an experiment w/ 2 outcomes

outcomes = {success, failure}

$P(\{\text{success}\}) = p \qquad 0 \le p \le 1$

$P(\{\text{failure}\}) = 1 - p$

We do n trials. What is the probability of at least 1 success?

$\{\text{at least 1 success}\}^c = \{\text{all failures}\} = F_1 F_2 \cdots F_n = \bigcap_{i=1}^{n} F_i$

$F_i = \{\text{failure on } i\text{th trial}\}$

$P(F_1 F_2 \cdots F_n) = P(F_1)P(F_2)\cdots P(F_n) = (1-p)^n$

Because trials are independent.

$$P(\text{at least 1 success}) = 1 - P(\text{all fail}) = 1-(1-p)^n$$

Do n trials

Q: $P(\text{exactly } k \text{ successes})$

$P(\text{a particular sequence of } k \text{ successes and } n-k \text{ failures})$

(eg. $P(SFSFSS) = p^4(1-p)^2$)

$$= p^k(1-p)^{n-k}$$

\# of sequence of k successe and $n-k$ $\binom{n}{k}$

$$P(\text{exactly } k \text{ success}) = \binom{n}{k} p^k (1-p)^{n-k}$$

$n=4$

$$SFSS = F_1^c F_2 F_3^c F_4^c$$

these are independent

$$FFSS = F_1 F_2 F_3^c F_4^c$$

these are independent

these intersections are mutually exclusive