Bayes's formula

S sample space E and F events

"conditioning on F"

$$E = EF \cup EF^c$$

Proof
$$E = ES = E(F \cup F^c)$$
$$= EF \cup EF^c$$

And EF and EF^c are mutually exclusive

$$P(E) = P(EF) + P(EF^c)$$

$$P(E|F) = \frac{P(EF)}{P(F)}$$

$$= P(E|F)P(F) + P(E|F^c)P(F^c)$$

"$P(E)$ is weighted average of $P(E|F)$ and $P(E|F^c)$"

Recall trick "reversing the conditional probability"

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B)}$$

Bayes's formula

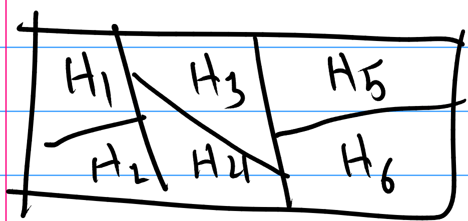
$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

More generally let $H_1, \dots, H_n$ be a collection of <u>mutually exclusive</u>, exh<u>aust</u>ive events

Mutually exclusive : $H_i H_j = \emptyset \quad (i \neq j)$

Exhaustive : $H_1 \cup H_2 \cup \cdots \cup H_n = S$

S



Consider H_i to be "hypotheses"

Now take any event E, which we regard as some "Evidence"

Bayes's formula
$$P(H_k | E) = \frac{P(E|H_k)\, P(H_k)}{\sum_{i=1}^{n} P(E|H_i)\, P(H_i)}$$

$P(E|H_i)$ - prob of evidence given hypothesis

$P(H_i)$ - "prior probability" of each hypothesis

Insurance company divides people into two class $\begin{cases} \text{accident prone} & A \\ \text{not accident prone} & A^c \end{cases}$

$P(B|A) = P($ prone person has accident in 1 year$) = .4$

$P(B|A^c) = P($ not accident prone has accident in 1 year$) = .2$

$P(A) = 30\%$ of population is accident prone

$S = \{$set of people$\}$ $A = \{$accident prone$\}$

$B = \{$people who have an accident in one year$\}$

Q: $P(B)$?

$$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c)$$
$$= (.4)(.3) + (.2)(.7) = .26$$

Q: Suppose a person has an accident B what is probability that this person is accident prone?

$$P(A|B) = \frac{P(B|A)\,P(A)}{\underbrace{P(B|A)P(A) + P(B|A^c)P(A^c)}_{.26}}$$

$$= \frac{(.4)(.3)}{.26} = \frac{6}{13}$$

We have some prior probabilities

$C = \{\text{cheating}\}$ $\quad C^c = \{\text{not cheating}\}$

$E =$ way that they played

$$P(C|E) = \frac{P(E|C)\,P(C)}{P(E|C)\,P(C) + P(E|C^c)\,P(C^c)}$$

Under what conditions is $P(C|E) > P(C)$

Need
$$\frac{P(E|C)}{P(E|C)\,P(C) + P(E|C^c)\,P(C^c)} > 1$$

$$P(E|C) > P(E|C)\,P(C) + P(E|C^c)\,P(C^c)$$

$$P(E|C)\,\cancel{[1-P(C)]} > P(E|C^c)\,\cancel{P(C^c)}$$

$$P(E|C) > P(E|C^c) \Rightarrow \text{increases confidence in } C$$

<u>Example</u> 3 cards

RR red on both sides
RB red / black
BB black on both sides

Card randomly selected and placed down on table, so we only get to see one side.

Suppose we see red facing up. what is prob that the other side is black?

Hypothesis = RR or RB or BB

Evidence = {see red} or {see black}

$$P(RB \mid \{\text{see red}\}) = \frac{P(\text{see red} \mid RB)\, P(RB)}{\left(\begin{array}{c} P(\text{see red} \mid RB)\, P(RB) + P(\text{see red} \mid RR)\, P(RR) \\ + P(\text{see red} \mid BB)\, P(BB) \end{array} \right)}$$

$$= \frac{\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}{\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{3}\right) + 0} = \frac{1}{3}$$