NAME: Solutions

EID:

M 362K Exam 4 **May 11, 2012** **Instructor: James Pascaleff**

Problem	Possible	Actual
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
Total	200	

INSTRUCTIONS:

- Do all work on these sheets.
- Show all work.
- No books, calculators, or other electronic devices.
- Two (2) two-sided 8.5 inch by 11 inch sheets of notes are permitted.

1. (20 pts) *Note: In all parts of this question, the answer is an integer, not a probability.*

   How many different linear arrangments of the letters A, B, C, D, E are there such that

   (a) (5 pts) A is the first in line?

   B–E in any order $\Rightarrow 4! = 24$

   (b) (5 pts) E is not the last in line?

   $5!$ total, $4!$ where E is last

   E not last $\Rightarrow 5! - 4! = 4 \cdot 4! = 96$

   (c) (5 pts) A comes before B?

   $\binom{5}{2} = 10$ ways to place A and B

   $3!$ ways to place rest $\Rightarrow \binom{5}{2} 3! = 10 \cdot 6 = 60$

   (d) (5 pts) C and D are next to each other?

   Treat C and D as one object $\Rightarrow 4!$ orders

   orders of C, D next to each other $\Rightarrow 2$

   $\Rightarrow 2 \cdot 4! = 48$

2. (20 pts) Recall the standard playing card deck: 52 cards divided into 4 suits $\spadesuit, \diamondsuit, \clubsuit, \heartsuit$, each suit containing 13 values A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K. We can denote a specific card by a pair of symbols: $3\diamondsuit$ = three of diamonds, $K\spadesuit$ = king of spades, and so on.

   In poker a *flush* is defined to be a set of *five cards of the same suit*, for example

$$\textbf{Example of FLUSH: } 5\spadesuit, 6\spadesuit, 9\spadesuit, J\spadesuit, A\spadesuit \tag{1}$$

   A *full house* is defined to be a set of five cards containing *three of a kind and a pair*, for example

$$\textbf{Example of FULL HOUSE: } 10\diamondsuit, 10\clubsuit, 10\heartsuit, 2\clubsuit, 2\spadesuit \tag{2}$$

   In a certain poker variant, I draw cards one at a time up to a total of 7, and then I select 5 of the seven cards to form my hand. After having already drawn 6 cards, I hold

$$\textbf{My CURRENT HAND: } 2\heartsuit, 2\spadesuit, 7\heartsuit, 7\clubsuit, K\heartsuit, A\heartsuit. \tag{3}$$

   With one more card (the seventh) left to draw, there are $52 - 6 = 46$ possible cards left (as 6 cards are already in my hand), and **I assume the seventh card is equally likely to be any of these 46 cards**. Note that **it is impossible to get the same card twice**, for example, I cannot get another $A\heartsuit$, but I can get $A\spadesuit, A\diamondsuit$, or $A\clubsuit$.

   (a) (5 pts) What is the probability that the seventh card gives me a flush? For example $9\heartsuit$ would give me five hearts, hence a flush.

To get flush: $13-4 = 9$ cards

3, 4, 5, 6, 8, 9, 10, J, Q of hearts

$$P(\text{flush}) = \frac{9}{46}$$

(b) (5 pts) What is the probability that the seventh card gives me a full house? For example 2♣ would give me three 2s and two 7s, hence a full house.

To get full house : 4 cards

2 clubs, 2 diamonds , 7 spades, 7 diamonds

$$P(\text{full house}) = \frac{4}{46}$$

(c) (5 pts) What is the probability that the seventh card gives me both a flush and a full house? (The flush and the full house may consist of different sets of five cards within the same seven-card hand.)

To get both, need heart and 2 or 7

$\Rightarrow$ 2 hearts or 7 hearts,

but both are already in my hand!

$$P(\text{both}) = 0$$

(d) (5 pts) What is the probability that the seventh card gives me either a flush or a full house or both?

$$P(\text{flush or full house})$$

$$= P(\text{flush}) + P(\text{full hous}) - P(\text{both})$$

$$= \frac{9}{46} + \frac{4}{46} - 0 = \frac{13}{46}$$

3. (20 pts) Suppose that two factories, A and B, make radios. The radios from factory A are defective with probability .1, while those from factory B are defective with probability .05.

   You buy a radio at the store, which is equally likely to have been made at either factory.

   (a) (10 pts) Suppose the radio you bought turns out to be defective. Given this knowledge, what is the conditional probability it was made in factory B?

A= made in factory A   D = defective
B= made in factory B

$$P(B|D) = \frac{P(D|B)P(B)}{P(D|A)P(A)+P(D|B)P(B)} = \frac{.05\left(\frac{1}{2}\right)}{.1\left(\frac{1}{2}\right)+.05\left(\frac{1}{2}\right)}$$

$$= \frac{.025}{.075} = \frac{1}{3}$$

   (b) (10 pts) You go back to the store to buy another radio, but the store owner tells you that all the radios in stock, including the defective one you previously bought, were made at the **same factory**, although she does not know which factory it is. Given his knowledge, what is the conditional probability that the second radio will also be defective?

D_2 = second is defective

$$P(D_2|D) = P(D_2|BD)P(B|D)+P(D_2|AD)P(A|D)$$

$$P(D_2|BD) = P(D_2|B) = .05$$

$$P(D_2|AD) = P(D_2|A) = .1$$

$P(B|D) = \frac{1}{3}$, $P(A|D) = \frac{2}{3}$ (from (a))

$$P(D_2|D) = .05\cdot\frac{1}{3} + .1\cdot\frac{2}{3} = \frac{.25}{3} = \frac{1}{12}$$

4. (20 pts) You have a chance to play a game that goes like this:

- Three fair ($P(H) = P(T) = .5$) coins are flipped.
- If the first coin is heads you win \$1, if tails you win nothing and lose nothing.
- If the second coin is heads you win \$2, if tails you win nothing and lose nothing.
- If the third coin is heads you win nothing and lose nothing, but if it is tails you lose \$4 (you must pay \$4, or "you win -4 dollars").

The game is summarized in this table:

	1st	2nd	3rd
H	+1	+2	0
T	0	0	−4

For example if you flip HHH, you win a total of $1 + 2 + 0 = 3$ dollars. If you flip HTT, you win $1 + 0 + (-4) = -3$ dollars, which means you lose 3 dollars.

What are your expected winnings in this game (the three flips combined)? Would you be willing to play it repeatedly, perhaps 1000 times?

HHH	3
HHT	−1
HTH	1
HTT	−3
THH	2
THT	−2
TTH	0
TTT	−4

all outcomes have prob $\frac{1}{8}$.

$$E[X] = \frac{3 - 1 + 1 - 3 + 2 - 2 + 0 - 4}{8}$$

$$= \frac{-4}{8} = -\frac{1}{2}$$

Not a good idea to play repeatedly because expectation is negative. Will lose money in the long run.

5. (20 pts) Suppose a basketball player is practicing shooting, and has a probability .95 of making each of his shots. Also assume that his shots are independent of one another.

(a) (7 pts) Let X be the number of shots made in 100 attempts. What is the probability mass function of X? What is $E[X]$?

Binomial $n=100$ $p=.95$

$$P\{X=i\} = \binom{100}{i}(.95)^i(.05)^{100-i}$$

$$E[X] = np = 100 \times .95 = 95$$

(b) (7 pts) Let Y be the number of shots made before the first miss. What is the probability that $Y > 50$?

Y is geometric with $p=.05$

$$P\{Y>50\} = \sum_{i=51}^{\infty}(.95)^{i-1}(.05) = .05 \cdot (.95)^{50}\frac{1}{1-.95}$$

$$= (.95)^{50}$$

(or $P\{Y>50\} = P\{$50 makes in a row$\}$ $= (.95)^{50}$)

(c) (6 pts) Using the Poisson distribution, approximate the probability that there are at most 2 misses in the first 100 attempts.

Z = # misses = Binomial with $n=100$, $p=.05$

$\lambda = np = 5$

$$P\{Z \le 2\} = P\{Z=0\} + P\{Z=1\} + P\{Z=2\}$$

$$= e^{-5} + 5e^{-5} + \frac{5^2}{2}e^{-5} = \frac{37}{2}e^{-5}$$

6. (20 pts) Let X be an exponential random variable with parameter λ, so that $E[X] = 1/\lambda$.

(a) (7 pts) What is the probability density function of X? Make sure your answer works for every real number x, both positive and negative.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

(b) (7 pts) Derive the cumulative distribution function of X from your answer to part (a). Again, make sure your answer works for any real number.

for $a < 0$ $\quad F(a) = \int_{-\infty}^{a} f(x)dx = \int_{-\infty}^{a} 0 dx = 0$

for $a \geq 0$ $\quad F(a) = \int_{-\infty}^{a} f(x)dx = \int_{0}^{a} \lambda e^{-\lambda x} dx$

$$= \left[-e^{-\lambda x}\right]_0^a = -e^{-\lambda a} - (-1) = 1 - e^{-\lambda a}$$

$$F(a) = \begin{cases} 1 - e^{-\lambda a} & a \geq 0 \\ 0 & a < 0 \end{cases}$$

(c) (6 pts) Using your answer to part (b), find the probability $P\{X > 1/\lambda\}$.

$$P\{X > 1/\lambda\} = 1 - P\{X \leq 1/\lambda\} = 1 - F(1/\lambda)$$

$$= 1 - \left(1 - e^{-\lambda(\frac{1}{\lambda})}\right) = e^{-1}$$

7. (20 pts) Suppose two random variables X and Y have joint density function

$$f(x,y) = \begin{cases} x+y & \text{if } 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(a) (15 pts) Find the density function of X and the density function of Y.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 (x+y)\,dy = \left[xy + \frac{y^2}{2}\right]_{y=0}^{y=1}$$

$$= x + \frac{1}{2} \quad \text{for } 0<x<1$$

$$f_X(x) = \begin{cases} x+\frac{1}{2} & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 (x+y)\,dx = y + \frac{1}{2}$$

$$f_Y(y) = \begin{cases} y+\frac{1}{2} & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

(b) (5 pts) Are X and Y independent? Why or why not?

$$f(x,y) = x+y \neq \left(x+\frac{1}{2}\right)\left(y+\frac{1}{2}\right) = f_X(x)f_Y(y)$$

so X and Y are <u>not</u> independent.

8. (20 pts) Ten cards numbered 1 through 10 are shuffled so that all orderings are equally likely, and they are turned up one at a time. We say that a match occurs if the first card is numbered 1, or if the second card is numbered 2, or ..., or the 10th card is numbered 10. That is to say, a match occurs if the ith card in the deck happens to be numbered i.

(a) (10 pts) Find the expected number of matches. *Hint:* Consider the random variables

$$X_i = \begin{cases} 1 & \text{if the } i\text{th card is a match,} \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

and consider the sum $\sum_{i=1}^{10} X_i$

$$X = \sum_{i=1}^{10} X_i = \#\text{matches}$$

$$E[X_i] = P\{X_i = 1\} = P\{\text{ith and is match}\} = \frac{9!}{10!} = \frac{1}{10}$$

$$E[X] = \sum_{i=1}^{10} E[X_i] = \sum_{i=1}^{10} \frac{1}{10} = 10\left(\frac{1}{10}\right) = 1.$$

(b) (10 pts) Find $\mathrm{Cov}(X_1, X_2)$, the covariance of X_1 and X_2. Use this to determine whether X_1 and X_2 are independent.

$$E[X_1] = E[X_2] = \frac{1}{10}$$

$$E[X_1 X_2] = P\{X_1 X_2 = 1\} = P\{\text{1st and 2nd are matches}\}$$

$$= \frac{8!}{10!} = \frac{1}{90}$$

$$\mathrm{Cov}(X_1, X_2) = E[X_1 X_2] - E[X_1]E[X_2]$$
$$= \frac{1}{90} - \frac{1}{10}\frac{1}{10} = \frac{1}{90} - \frac{1}{100} = \frac{1}{900}$$

Since $\mathrm{Cov}(X_1, X_2) \neq 0$, X_1 and X_2 are not independent.

9. (20 pts) Suppose X and Y are independent normal random variables with the same mean $\mu = 0$ and the same variance σ^2 (which may be anything). Then the pair (X, Y) gives the coordinates of a random point in the plane. Let $A = A(X, Y)$ denote the area of the circle centered at the origin passing through (X, Y). (That is to say, the circle has center $(0, 0)$ and radius $R = \sqrt{X^2 + Y^2}$.) Compute the expected value of A.

Hint: This problem can be solved without explicitly using the density function of the normal random variable.

$$A(X,Y) = \pi R^2 = \pi(X^2+Y^2)$$

$$E[A] = E[\pi(X^2+Y^2)] = \pi E[X^2] + \pi E[Y^2]$$

$$E[X^2] = \mathrm{Var}(X) + (E[X])^2 = \sigma^2 \cancel{+\mu^2} = \sigma^2$$

$E[Y^2] = \sigma^2$ similarly

so $E[A] = \pi\sigma^2 + \pi\sigma^2 = 2\pi\sigma^2$

10. (20 pts) Suppose that $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of independent identically distributed random variables with mean 1 and variance 1600, and assume that these variables are non-negative: $P\{X_i \geq 0\} = 1$.

Let Y be the sum of the first 100 variables: $Y = \sum_{i=1}^{100} X_i$.

(a) (10 pts) What does Markov's inequality tell you about the probability $P\{Y \geq 900\}$?

$$P\{Y \geq 900\} \leq \frac{E[Y]}{900} = \frac{100}{900} = \frac{1}{9} \approx 11.1\%$$

since $E[Y] = \sum_{i=1}^{100} E[X_i] = \sum_{i=1}^{100} 1 = 100$

(b) (10 pts) Use the central limit theorem to approximate the probability $P\{Y \geq 900\}$.

$$\mu = 1, \quad \sigma = \sqrt{1600} = 40, \quad n = 100$$

$$P\{Y \geq 900\} = P\left\{\frac{Y - n\mu}{\sigma\sqrt{n}} \geq \frac{900 - n\mu}{\sigma\sqrt{n}}\right\}$$

$$= P\left\{\frac{Y - 100}{40 \cdot 10} \geq \frac{900 - 100}{40 \cdot 10}\right\} = P\left\{\frac{Y - 100}{400} \geq 2\right\}$$

Assuming (as we are instructed to) that the central limit theorem gives a good approximation, this is

$$\approx P\{Z \geq 2\} = 1 - P\{Z \leq 2\} = 1 - .9772 = .0228$$

$$\approx 2.3\%$$