NAME: Solutions

EID:

M 362K Exam 3 **April 20, 2012** **Instructor: James Pascaleff**

INSTRUCTIONS:

- Do all work on these sheets.
- Show all work.
- No books, calculators, or other electronic devices.
- One two-sided 8.5 inch by 11 inch sheet of notes is permitted.

Problem	Possible	Actual
1	15	
2	20	
3	25	
4	20	
5	20	
Total	100	

1. (15 points) Suppose that a continuous random variable X has density function given by

$$f(x) = \begin{cases} x & \text{if } 0 < x < \sqrt{2} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) Find the cumulative distribution function of X. Make sure your formulas cover all cases for what x can be.

for $0 < a < \sqrt{2}$ $\quad F(a) = \int_0^a x\,dx = \frac{a^2}{2}$

for $a \leq 0$ $\quad F(a) = 0$, for $a \geq \sqrt{2}$, $F(a) = 1$

$$F(a) = \begin{cases} 0 & a \leq 0 \\ a^2/2 & 0 < a < \sqrt{2} \\ 1 & \sqrt{2} \leq a \end{cases}$$

(b) Find the expectation of X.

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\sqrt{2}} x \cdot x\,dx = \left[\frac{x^3}{3}\right]_0^{\sqrt{2}} = \frac{(\sqrt{2})^3}{3} = \frac{2\sqrt{2}}{3}$$

(c) Find the variance of X.

$$E[X^2] = \int_0^{\sqrt{2}} x^2 \cdot x\,dx = \left[\frac{x^4}{4}\right]_0^{\sqrt{2}} = \frac{(\sqrt{2})^4}{4} = \frac{4}{4} = 1$$

$$\mathrm{Var}(X) = 1 - \left(\frac{2\sqrt{2}}{3}\right)^2 = 1 - \frac{4 \cdot 2}{9} = \frac{1}{9}$$

2. (20 points) Suppose that we flip a fair coin ($p = .5$) one hundred times. Let X denote the number of heads. What is the probability that we get heads between 43 and 57 times, that is, $43 \le X \le 57$? Use the normal approximation to the binomial distribution, with the continuity correction. Your answer should be a decimal number. (Note: If done correctly the calculations are not too hard to do by hand, though you will need to use the table.)

$P\{43 \le X \le 57\} = P\{42.5 < X < 57.5\}$ continuity correction

$\mu = np = 100 \cdot .5 = 50$, $\sigma^2 = np(1-p) = 100 \cdot .5 \cdot .5 = 25$

$\sigma = 5$

$$P\{42.5 < X < 57.5\} = P\left\{\frac{42.5-50}{5} < Z < \frac{57.5-50}{5}\right\}$$

$$= P\{-1.5 < Z < 1.5\} = \Phi(1.5) - \Phi(-1.5)$$

since $\Phi(-x) = 1 - \Phi(x)$: $= \Phi(1.5) - 1 + \Phi(1.5)$

$$= 2\Phi(1.5) - 1$$

$$= 2(.9332) - 1 = \boxed{.8664}$$

$$\begin{array}{r} .9332 \\ \times \quad 2 \\ \hline 1.8664 \end{array}$$

3. (25 points) Suppose that a certain type of car can be driven X miles between when it comes out of the factory until it breaks down completely, where X is an exponential random variable with $\lambda = 1/200000 = 5 \times 10^{-6}$.

   (a) (10 points) Suppose you are considering buying one of these cars used, with $50000 = 5 \times 10^4$ miles already on it. What is the probability that you can drive the car another $100000 = 10^5$ miles before it breaks down completely?

For X exponential : $P\{X>t\} = e^{-\lambda t}$

By memorylessness:

$P\{X > 10^5 + 5\times10^4 \mid X > 5\times10^4\} = P\{X > 10^5\}$

$= e^{-(5\times10^{-6})(10^5)} = e^{-.5}$

(b) (10 points) Suppose that a car of this type is repeatedly bought and sold. Each owner drives the car for $100000 = 10^5$ miles, then sells it, and the next owner buys it. This process continues until the car breaks down completely. What is the probability that the car breaks down while it is in the possession of the n-th owner?

Break down on nth owner: $P\{(n-1)10^5 < X < n \cdot 10^5\}$

$$= P\{X > (n-1)10^5\} - P\{X > n \cdot 10^5\}$$

$$= e^{-\lambda(n-1)10^5} - e^{-\lambda n 10^5} = e^{-(.5)(n-1)} - e^{-(.5)n}$$

OR, interpreting the "while clause" as a conditional prob.

$$P\{\text{break down on nth} \mid \text{nth buys it}\} = P\{X < n \cdot 10^5 \mid X > (n-1)10^5\}$$

$$= 1 - P\{X > n \cdot 10^5 \mid X > (n-1)10^5\} = 1 - P\{X > 10^5\}$$

$$= 1 - e^{-.5}$$

(c) (5 points) Let N be a random variable denoting the number of owners the car has until it breaks down. Show that N is a geometric random variable, and identify the value of the parameter p.

$$p(n) = P\{N = n\} = e^{-(.5)(n-1)} - e^{-(.5)n} = e^{-(.5)(n-1)}\left(1 - e^{-(.5)}\right)$$

$$= \left(e^{-(.5)}\right)^{n-1}\left(1 - e^{-(.5)}\right) = (1-p)^{n-1} p$$

where $\boxed{p = \left(1 - e^{-(.5)}\right)}$

Thus N is geometric.

4. (20 points) Suppose that X and Y are independent random variables, and that X is binomial with parameters (n, p), while Y is binomial with parameters (m, p). The value of p is the same in both cases.

(a) (15 points) Find the probability mass function of $X + Y$, and identify what type of random variable $X + Y$ is. *Hint:* you will need to use the identity

$$\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} \tag{2}$$

$$p_{X+Y}(k) = \sum_{i=0}^{k} p_X(i)\, p_Y(k-i)$$

$$= \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i} \binom{m}{k-i} p^{k-i} (1-p)^{m-k+i}$$

$$= \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} p^k (1-p)^{n+m-k}$$

$$= \left[\sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i}\right] p^k (1-p)^{n+m-k} = \binom{n+m}{k} p^k (1-p)^{n+m-k}$$

So $X+Y$ is Binomial with parameters $(n+m, p)$

(b) (5 points) Give an interpretation of your answer in terms of the process of $n+m$ Bernoulli trials.

We do $n+m$ Bernoulli trials:

Let X = # successes in first n trials

Y = # successes in last m trials

Z = Total # of successes in $n+m$ trials

Then X is binomial w/ (n,p), Y binomial w/ (m,p)

Z binomial w/ $(n+m,p)$, and clearly $Z = X+Y$

5. (20 points) The process of people entering a particular store is a Poisson process with average rate $\lambda = 3$ people per hour. Consider the random variables

- X = time (in hours) from when the store opens until the first person enters the store.
- Y = time (in hours) from when the first person enters until the second person enters.
- $X + Y$ = time (in hours) from when the store opens until the second person enters.

So, as was shown in lecture, X and Y are exponential variables with $\lambda = 3$, and $X + Y$ is a Gamma variable with $\lambda = 3$ and $\alpha = 2$. Furthermore, X and Y are independent.

Given the condition $X + Y = 1$, what is the conditional probability density function of X? That is, find the function $f(x \mid X + Y = 1)$ such that

$$P\{X \le a \mid X + Y = 1\} = \int_{-\infty}^{a} f(x \mid X + Y = 1)\, dx \tag{3}$$

Hint: There are several approaches to this problem. One possibility is to try to figure out the conditional density of $X + Y$ given X, and then "reverse the order of conditioning." Another is to try to compute $P\{X \le a \mid 1 - \epsilon < X + Y < 1 + \epsilon\}$, and then take ϵ to 0. There are also other approaches.

Given $X = x$, $P\{X+Y \approx t \mid X = x\} = P\{Y \approx t - x\}$

so $f_{X+Y|X}(t|x) = f_Y(t-x) = \begin{cases} \lambda e^{-\lambda(t-x)} & t-x>0 \\ 0 & \text{other} \end{cases}$

$f_{X+Y|X}(t|x) = \dfrac{f_{X+Y,X}(t,x)}{f_X(x)} \Rightarrow f_{X+Y,X}(t,x) = f_X(x) f_Y(t-x)$

$f_{X+Y,X}(t,x) = \lambda e^{-\lambda x} \lambda e^{-\lambda(t-x)} = \lambda^2 e^{-\lambda t}$ valid if $\begin{matrix} x>0 \\ t-x>0 \end{matrix}$

$f_{X|X+Y}(x|t) = \dfrac{f_{X+Y,X}(t,x)}{f_{X+Y}(t)} = \dfrac{\lambda^2 e^{-\lambda t}}{\lambda e^{-\lambda t}(\lambda t)} = \dfrac{1}{t}$ valid if $\begin{matrix} x>0 \\ t-x>0 \end{matrix}$

$f(x|X+Y=1) = f_{X|X+Y}(x|1) = 1$ valid if $\begin{matrix} x>0 \\ 1-x>0 \end{matrix}$ i.e. $0<x<1$

$f(x|X+Y=1) = 0$ otherwise.

So we have a uniform distribution on $(0,1)$.

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$$(*)\ P\{X \le a \mid 1-\varepsilon < X+Y < 1+\varepsilon\} = \frac{P\{X \le a\,,\ 1-\varepsilon < X+Y < 1+\varepsilon\}}{P\{1-\varepsilon < X+Y < 1+\varepsilon\}}$$

$$P\{X \le a\,,\ 1-\varepsilon < X+Y < 1+\varepsilon\} = \int_{-\infty}^{a}\int_{1-x-\varepsilon}^{1-x+\varepsilon} f_{X,Y}(x,y)\,dy\,dx = \int_{0}^{a}\int_{1-x-\varepsilon}^{1-x+\varepsilon} \lambda e^{-\lambda x}\lambda e^{-\lambda y}\,dy\,dx$$

$$= \int_0^a \lambda e^{-\lambda x}\left[e^{-\lambda(1-x-\varepsilon)} - e^{-\lambda(1-x+\varepsilon)}\right]dx = \int_0^a \lambda\left[e^{-\lambda(1-\varepsilon)} - e^{-\lambda(1+\varepsilon)}\right]dx$$

$$= a\lambda e^{-\lambda}\left[e^{\lambda\varepsilon} - e^{-\lambda\varepsilon}\right]$$

$$P\{1-\varepsilon < X+Y < 1+\varepsilon\} \simeq 2\varepsilon f_{X+Y}(1) = 2\varepsilon\lambda e^{-\lambda(1)}\lambda(1) = 2\varepsilon\lambda^2 e^{-\lambda}$$

$$(*) \approx \frac{a\lambda e^{-\lambda}\left[e^{\lambda\varepsilon} - e^{-\lambda\varepsilon}\right]}{2\varepsilon\lambda^2 e^{-\lambda}} = \frac{a\left[1+\lambda\varepsilon+\cdots - 1+\lambda\varepsilon-\cdots\right]}{2\lambda\varepsilon} = a$$

This is valid as long as $0 < a < 1$. So we have uniform dist on $(0,1)$

$$\Rightarrow f(x \mid X+Y=1) = \begin{cases} 1 & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$