NAME: Solutions

EID:

M 362K Exam 2 March 23, 2012 Instructor: James Pascaleff

INSTRUCTIONS:

- Do all work on these sheets.
- Show all work.
- No books, calculators, or other electronic devices.
- One two-sided 8.5 inch by 11 inch sheet of notes is permitted.
- Answers do not necessarily need to be simplified: $4\binom{19}{3}/22!$ is perfectly fine.

Problem	Possible	Actual
1	20	
2	20	
3	20	
4	25	
5	15	
Total	100	

1. (20 points) Suppose two four-sided dice, with sides numbered 1 through 4 are rolled. (Such a die has the shape of a tetrahedron or triangular pyramid.) Let X denote the sum of the dice.

   (a) (5 pts.) What are the possible values of X?

   $X = 2, 3, 4, 5, 6, 7, 8$

   (b) (10 pts.) Find the probability mass function of X.

X	possibilities	prob
2	11	$1/16 = P(X=2)$
3	12, 21	$2/16 = P(X=3)$
4	13, 22, 31	$3/16 = P(X=4)$
5	14, 23, 32, 41	$4/16 = P(X=5)$
6	24, 33, 42	$3/16 = P(X=6)$
7	34, 43	$2/16 = P(X=7)$
8	44	$1/16 = P(X=8)$

   (c) (5 pts.) Write out, but *do not evaluate*, a sum for $E[X]$, the expectation of X. (Just write out the terms, you don't have to simplify).

$$E[X] = \sum_{i=2}^{8} i\,P(X=i) = 2\frac{1}{16} + 3\frac{2}{16} + 4\frac{3}{16} + 5\frac{4}{16} + 6\frac{3}{16} + 7\frac{2}{16} + 8\frac{1}{16}$$

2. (20 points)

   Part 1 (10 points): A certain variant of the game Three-card Monte is played as follows: You wager \$1.00. The dealer has three cards, a Queen, a seven, and a two. They are shuffled and laid face down. Your challenge is to guess which card is the Queen. If you guess incorrectly, you lose your wager. If you guess correctly, you win \$1.50.

   What is your expected winnings in this game? Would you be willing to play this game 100 times? Explain your answer in terms of expectation (one sentence is enough).

$P(\text{win}) = \frac{1}{3}$ $\quad P(\text{lose}) = \frac{2}{3}$

win $\Rightarrow X = 1.50$ $\quad$ lose $\Rightarrow X = -1.00$

$$E[X] = 1.50\left(\tfrac{1}{3}\right) + (-1.00)\tfrac{2}{3} = \frac{-.5}{3} = -\frac{1}{6} \approx -.17$$

No would not play 100 times because expectation is <u>negative</u>. In the long run we lose 17¢ per game.

Part 2 (10 points): Suppose that the dealer offers a modification of the rules: If you pay the dealer C cents, he will tell you which of the cards is the seven before you make your guess. Note that you cannot win back these C cents if you guess correctly; you either lose \$1.00 or win \$1.50, but you have more information.

What is your expected winnings if you buy the extra information and play the game this way? What is your expected "net" winnings taking into account the price paid for the information? For what values of C (the price of the extra information) would you be willing to play the game 100 times? Explain (one sentence is enough).

$P(\text{win} \mid \text{info}) = \frac{1}{2}$ $\quad P(\text{lose} \mid \text{info}) = \frac{1}{2}$

so $E[X] = (1.50)\frac{1}{2} + (-1.00)\frac{1}{2} = \frac{.5}{2} = .25$

Net winnings is $.25 - \frac{C}{100}$ (C is in cents)

As long as $C < 25$ cents, expected net winnings is positive, so I would play.

3. (20 points) Suppose that X is a random variable having possible values 0 and 1. Suppose that

- $P(X = 0) = 2P(X = 1)$ (the value ~~1~~ 0 is twice as likely as the value ~~0~~ 1)
- $E[X] = 1/3$.

(a) (10 pts.) Find the probability mass function of X.

$$\frac{1}{3} = E[X] = 0\,P(X=0) + 1\,P(X=1) = P(X=1)$$

so $P(X=1) = \frac{1}{3}$

$$P(X=0) = 2P(X=1) = \frac{2}{3}$$

(b) (10 pts.) Find the variance of X.

$$E[X^2] = 0^2 P(X=0) + 1^2 P(X=1) = P(X=1) = \frac{1}{3}$$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \frac{1}{3} - \left(\frac{1}{3}\right)^2 = \frac{2}{9}$$

4. (25 points) Suppose an NBA basketball team has a probability .6 of winning each of its home games, and probability .5 of winning each of its away games. All games are independent. The regular season consists of 82 games, 41 home and 41 away.

   (a) (5 pts.) Find an exact expression for the probability that the team wins exacty 22 home games.

Binomial $n = 41$ $p = .6$

$$P(\text{win } 22) = \binom{41}{22} (.6)^{22} (.4)^{19}$$

   (b) (5 pts.) What is the probability that the first home win occurs on the fifth home game?

geometric $p = .6$

$$P(\text{first win on } 5\text{th}) = (1-p)^{5-1} p = (.4)^4 (.6)$$

(c) (10 pts.) Suppose that a three-game series is played by this team, in the pattern

home, away, home.

What is the probability that the team wins exactly two out of these three games? (Note: all three games will be played no matter what.)

$P(WWL) = (.6)(.5)(.4)$

$P(WLW) = (.6)(.5)(.6)$

$P(LWW) = (.4)(.5)(.6)$

$P(\text{win 2 of 3}) = \text{sum} \uparrow = 2(.6)(.5)(.4) + (.6)^2(.5)$

(5 pts.) Suppose a different (terrible) NBA team has probability .05 of winning each game (regardless of whether the game is home or away). Find an *approximate* expression for the probability that this team wins k regular-season games. You will not get credit if your answer involves binomial coefficients.

Poisson approx to binomial $\quad n = 82 \quad p = .05$

$\lambda = np = 82(.05) = 4.1$

$P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!} = e^{-4.1}\frac{(4.1)^k}{k!}$

5. (15 points) Suppose that people wandering in a shopping mall enter the shoe store at an average rate of 10 per hour. Suppose that the entry of people at the store may be modeled by a Poisson process.

(a) (5 pts.) Over the course of an hour, what is the probability that at least 3 people enter the shoe store?

Poisson w/ $\lambda = 10$

$$P(N \geq 3) = 1 - P(N=0) - P(N=1) - P(N=2)$$

$$= 1 - e^{-10} - 10e^{-10} - \frac{(10)^2}{2}e^{-10} = 1 - 61e^{-10}$$

(b) (5 pts.) For how many hours must the store be open in order for the expected number of people entering the shoe store in that time period to be 100? What is the probability that 0 people enter the store during that time period?

$$E[N(t)] = \lambda t = 10t$$

so in 10 hours, expected # = 100

$$P(N(10) = 0) = e^{-\lambda t}\frac{(\lambda t)^0}{0!} = e^{-10 \cdot 10} = e^{-100}$$

(c) (5 pts.) Suppose we know that starting from any particular moment in time, the amount of time we must wait for the next person to enter the store is a continuous random variable with probability density function $f(x)$:

$$f(x) = \begin{cases} 10e^{-10x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Here x is measured in hours. Find the probability that we must wait more than 6 minutes ($= 0.1$ hours) for the next person to enter the store.

$$P(X > .1) = \int_{.1}^{\infty} 10e^{-10x}\,dx$$

$$= \left[-e^{-10x}\right]_{.1}^{\infty} = \left(\lim_{x\to\infty} -e^{-10x}\right) + e^{-10\cdot(.1)}$$

$$= 0 + e^{-1} = e^{-1}$$