

NAME: *Solutions*

EID:

M 362K Exam 1 February 17, 2012

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INSTRUCTIONS:

- Do all work on these sheets.
- Show all work.
- No books, notes, calculators, or other electronic devices.
- Answers do not necessarily need to be simplified: $4\binom{19}{3}/22!$ is perfectly fine.

Problem	Possible	Actual
1	15	
2	15	
3	15	
4	15	
5	20	
6	20	
Total	100	

1. (15 points) Consider 2 classes, the first with N students, and the second with M students. From this group of $N+M$ students, a pair of students (that is, an unordered set of 2 students) is to be selected.

- (a) How many choices are possible?

$$\binom{N+M}{2}$$

- (b) How many choices are possible if both students come from the first class?

$$\binom{N}{2}$$

- (c) How many choices are possible if both students come from the second class?

$$\binom{M}{2}$$

- (d) How many choices are possible if the pair includes one student from each class?

$$N \cdot M$$

- (e) Write down a combinatorial identity relating the answers to the previous 4 parts.

$$\binom{N+M}{2} = \binom{N}{2} + \binom{M}{2} + N \cdot M$$

2. (15 points) The lottery in the city of Probandia works like this: 100 balls numbered 0-99 are placed in an urn, and 5 balls are withdrawn, giving an **ordered** sequence of 5 numbers. Once balls are drawn they are **not replaced** before the next are drawn. Citizens buy tickets with 5 numbers of their choosing. The jackpot is awarded for matching all 5 numbers in the right order.

- (a) How many possible outcomes of the lottery drawing are there?

$$100 \cdot 99 \cdot 98 \cdot 97 \cdot 96 = \binom{100}{5} \cdot 5!$$

- (b) Mrs. Bernoulli plays 5 numbers corresponding to the days-of-the-month on which each of her 5 children were born (her children were born on different days). She buys several tickets, one for each of permutations of these 5 numbers. How many tickets does she buy? What is her chance of winning the jackpot?

$$\# \text{ tickets} = 5!$$

$$P(\text{winning}) = \frac{5!}{100 \cdot 99 \cdot 98 \cdot 97 \cdot 96}$$

- (c) A separate prize is awarded for getting the correct set of 5 numbers, but not necessarily in the right order. What is the probability that any given ticket will win this prize? What is Mrs. Bernoulli's probability of winning this prize?

$\binom{100}{5}$ outcomes for the set of numbers

$$P(\text{winning}) = \binom{100}{5}^{-1}$$

for Mrs. B also $\binom{100}{5}^{-1}$ since all her tickets give same subset.

3. (15 points) In a standard deck of 52 playing cards, the spades and clubs are black, while the hearts and diamonds are red. Hence there are 26 cards of each color. What is the probability that a random 5-card hand will contain cards of both colors? Explain your reasoning.

$$\begin{array}{lll}
 1 \text{ red} & 4 \text{ black} & \binom{26}{1} \binom{26}{4} \\
 2 \text{ red} & 3 \text{ black} & \binom{26}{2} \binom{26}{3} \\
 3 \text{ red} & 2 \text{ black} & \binom{26}{3} \binom{26}{2} \\
 4 \text{ red} & 1 \text{ black} & \binom{26}{4} \binom{26}{1}
 \end{array}$$

$$\text{So } P = \left[\binom{26}{1} \binom{26}{4} + \binom{26}{2} \binom{26}{3} + \binom{26}{3} \binom{26}{2} + \binom{26}{4} \binom{26}{1} \right] \binom{52}{5}^{-1}$$

$$\text{OR} \quad \text{All black} \quad \frac{\binom{26}{5}}{\binom{52}{5}}$$

$$\text{All red} \quad \frac{\binom{26}{5}}{\binom{52}{5}}$$

$$P(\text{at least one of each}) = 1 - \frac{\binom{26}{5}}{\binom{52}{5}} - \frac{\binom{26}{5}}{\binom{52}{5}}$$

4. (15 points) Suppose we roll a fair six-sided die 10 times. What is the probability that we roll a 6 at least 1 time, but not more than 5 times? Explain your reasoning.

$$1 \text{ 6's: } \binom{10}{1} \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^9$$

$$4 \text{ 6's } \binom{10}{4} \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^6$$

$$2 \text{ 6's: } \binom{10}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^8$$

$$5 \text{ 6's } \binom{10}{5} \left(\frac{1}{6}\right)^5 \left(\frac{5}{6}\right)^5$$

$$3 \text{ 6's } \binom{10}{3} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^7$$

$$P = \text{sum of these 5 terms} = \sum_{i=1}^5 \binom{10}{i} \left(\frac{1}{6}\right)^i \left(\frac{5}{6}\right)^{10-i}$$

5. (20 points) An urn contains 2 dice. One is fair, and the other is loaded. The fair die will land on any of the numbers 1 through 6 with equal probability $1/6$. The loaded die will land on 6 with probability $1/2$, and will land on each of the other numbers 1 through 5 with probability $1/10$. One of the dice is selected at random (so each die has probability $1/2$ of being selected).

- (a) What is the probability that the randomly chosen die will roll a six? Show the work that leads to your answer.

$$\begin{aligned}
 P(6) &= P(6 | \text{fair}) P(\text{fair}) + P(6 | \text{loaded}) P(\text{loaded}) \\
 &= \left(\frac{1}{6}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{12} + \frac{1}{4} = \frac{1}{3}
 \end{aligned}$$

- (b) Suppose the die is rolled 3 times, and comes up (6,6,3). What is the probability that the die is fair? Show the work that leads to your answer.

$$P(\text{fair} | (663)) = \frac{P(663 | \text{fair}) P(\text{fair})}{P(663 | \text{fair}) P(\text{fair}) + P(663 | \text{loaded}) P(\text{loaded})}$$

$$P(663 | \text{fair}) = \left(\frac{1}{6}\right)^3 = \frac{1}{216} \quad P(663 | \text{loaded}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{10} = \frac{1}{40}$$

$$\text{so get } \frac{\frac{1}{216} \cdot \frac{1}{2}}{\frac{1}{216} \cdot \frac{1}{2} + \frac{1}{40} \cdot \frac{1}{2}} = \frac{\frac{1}{216}}{\frac{1}{216} + \frac{1}{40}} = \frac{40}{256}$$

6. (20 points) Given two events E and F , the *symmetric difference* $E\Delta F$ is defined to be

$$E\Delta F = EF^c \cup E^cF \quad (1)$$

The symmetric difference corresponds to the *exclusive or* operation:

$$E\Delta F = E \text{ occurs or } F \text{ occurs but not both} \quad (2)$$

(a) Using a Venn diagram and the axioms of probability, prove that

$E\Delta F = I \cup III$

$$P(E\Delta F) = P(E) + P(F) - 2P(EF) \quad (3)$$

$$P(E) = P(I) + P(II)$$

$$P(F) = P(III) + P(II)$$

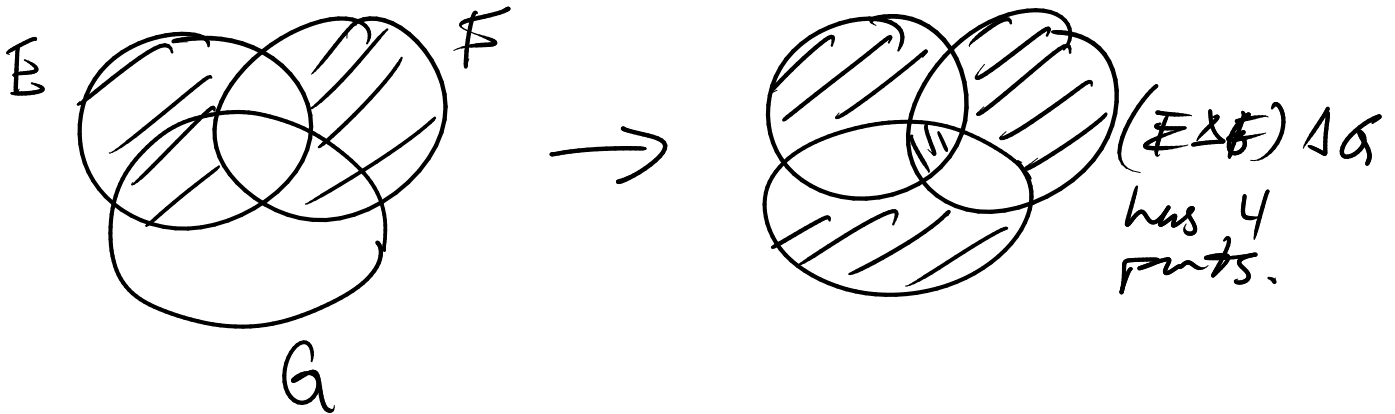
$$P(EF) = P(II)$$

$$P(E\Delta F) = P(I) + P(III)$$

$$= P(E) + P(F) - 2P(II)$$

$$= P(E) + P(F) - 2P(EF)$$

- (b) Draw a Venn diagram of $(E \Delta F) \Delta G$, where E , F , and G are three events, and find a formula for $P((E \Delta F) \Delta G)$ in terms of the probabilities of E, F, G, EF, EG, FG , and EFG . You don't need to formally prove your formula.



$$\begin{aligned}
 P((E \Delta F) \Delta G) &= P(E) + P(F) + P(G) \\
 &\quad - 2P(EF) - 2P(FG) - 2P(EG) \\
 &\quad + 4P(EFG)
 \end{aligned}$$