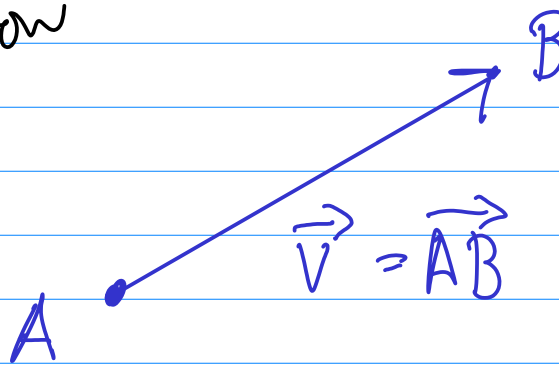
Vectors

Mantra: A vector is a "quantity" that has a magnitude and a direction. This is drawn using an arrow

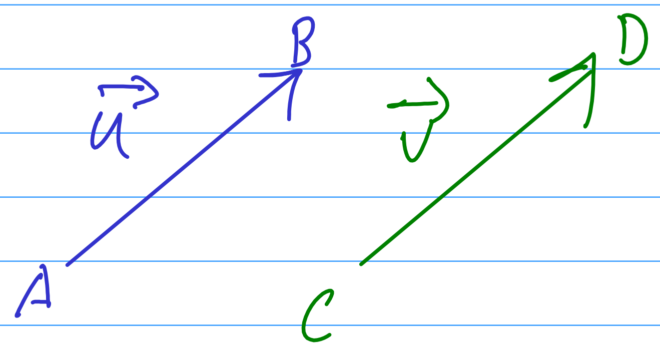


A = initial point

B = terminal point

"magnitude" = length of the arrow.

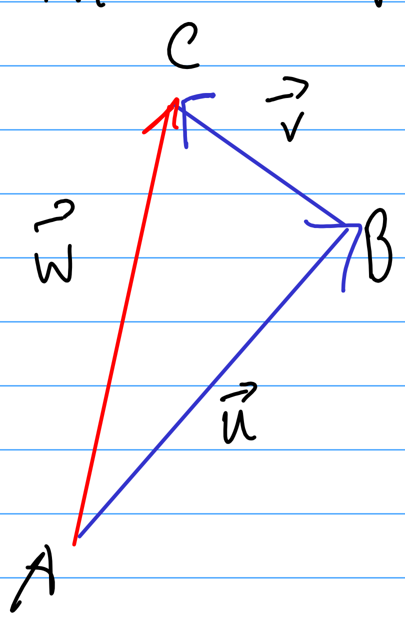
Vectors with the same length and direction are regarded as equal



$\vec{u} = \vec{v}$

The vector with zero magnitude is denoted by $\vec{0}$. It's direction is undefined.

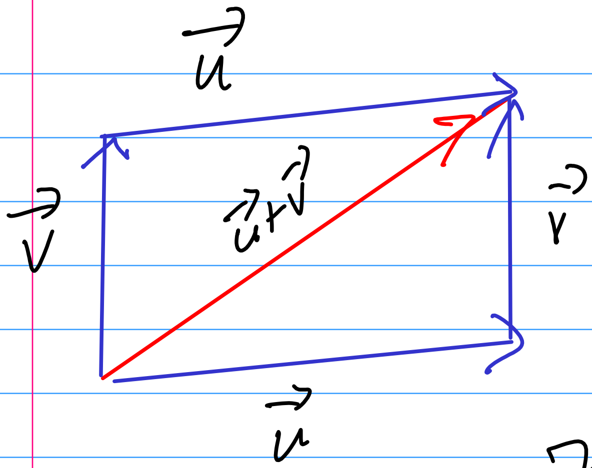
Addition = Triangle rule



$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

$\vec{u} + \vec{v} = \vec{w}$

add vectors tip-to-tail.



The parallelogram shows that we get the same answer using either triangle.

Scalar multiplication $=$ change the magnitude but not direction.

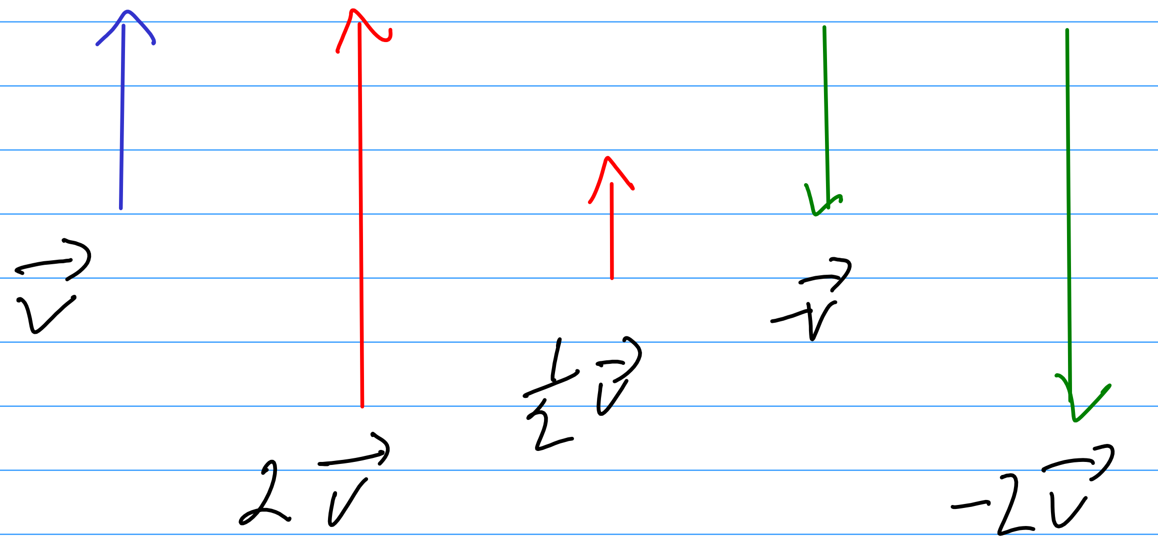
* If c is a real number, then the vector $c \cdot \vec{v}$ has length $|c| \cdot$ length of $\vec{v}$

Direction of $c\vec{v}$:

If $c > 0$, same direction as $\vec{v}$

If $c < 0$, opposite direction as $\vec{v}$

If $c = 0$, $c\vec{v} = 0$.



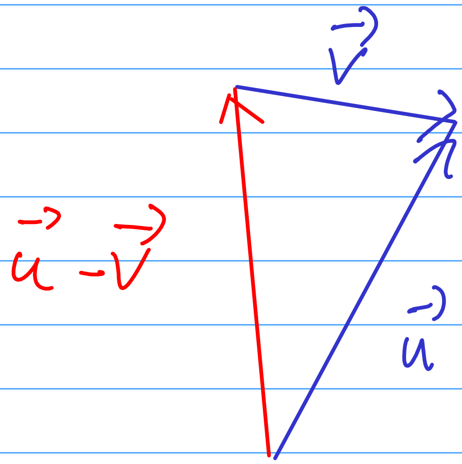
So scalar multiplication "rescales" the vector, hence the name.

Definitions:

* Two vectors are parallel if they are scalar multiples of each other
* the negative of $\vec{v}$ is $-\vec{v} = (-1)\vec{v}$.
* subtraction is adding the negative.

$$\vec{u} - \vec{v} = u + (-\vec{v})$$

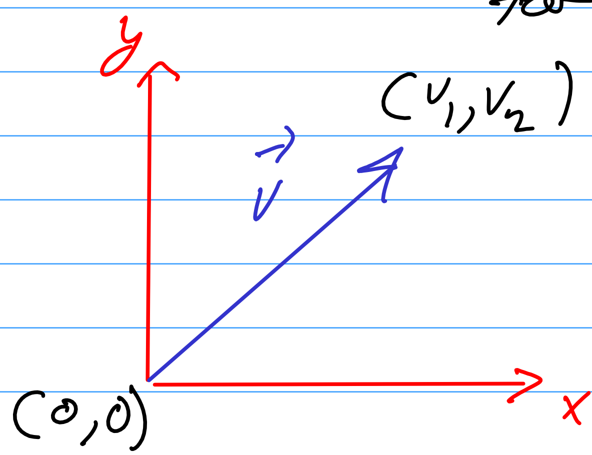
Picture of subtraction



evidently

$(\vec{u} - \vec{v}) + \vec{v} = \vec{u}$

Components: a coordinate system for vectors.



put tail at the origin, coordinates of the tip are the components of $\vec{v}$.

Tail = (0, 0) Tip = (v_1, v_2)

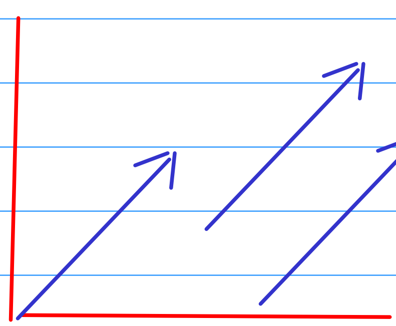
$\vec{v} = \langle v_1, v_2 \rangle$

Angle brackets = components of vector

Same notation for 3-dimensional vectors $\vec{v} = \langle v_1, v_2, v_3 \rangle$

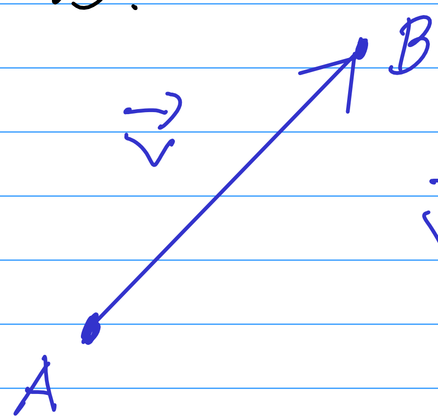
What is the relationship between vectors and points? both have coordinates/components, but these mean slightly different things

* A point has a definite position, but no magnitude or direction.
* A vector has a magnitude and direction, but no particular position



These are all the same as vectors.

* A vector can be thought of as the difference of two points.



$\vec{v} = $ "$B - A$"

In fact, if the coordinates of A, B are

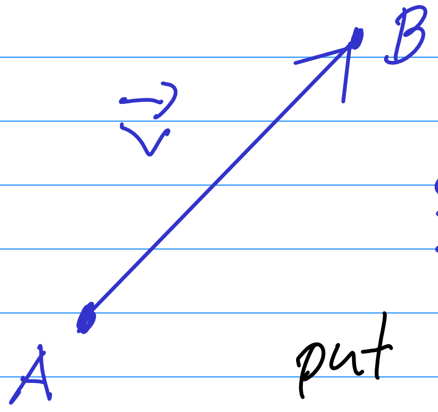
$A = (x_1, y_1, z_1)$

$B = (x_2, y_2, z_2)$

Then the components of $\vec{v}$ are

$$\vec{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

* In the same sense, the sum of a point and a vector is a point



$$B = \text{"} A + \vec{v} \text{"}$$

put tail at A, then B is the tip.

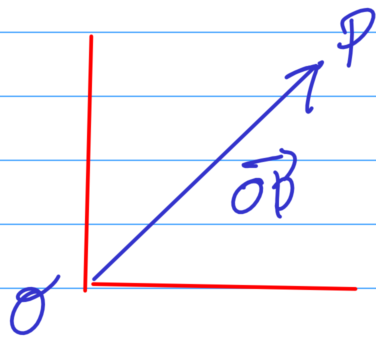
In fact, if coords of $A = (x, y, z)$ and components of $\vec{v} = \langle v_1, v_2, v_3 \rangle$ then the coordinates of B are

$$B = (x + v_1, \; y + v_2, \; z + v_3).$$

Now we see that any point P can be converted into a vector, by taking $\overrightarrow{OP}$ = vector from O to P

$= $ "$P - O$"

If coordinates of P are (x, y, z), the components of $\overrightarrow{OP}$ are $\langle x, y, z \rangle$.



This is called the position vector of P.

We can write all of the operations on vectors in components

* Magnitude or length $\|\vec{v}\|$

2d $\vec{v} = \langle v_1, v_2 \rangle$

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$$

3d $\vec{v} = \langle v_1, v_2, v_3 \rangle$

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

* addition, subtraction, scalar multiplication

component by component

$\vec{v} = \langle v_1, v_2 \rangle$ $\quad \vec{u} = \langle u_1, u_2 \rangle$

$$\vec{v} + \vec{u} = \langle v_1 + u_1, v_2 + u_2 \rangle$$

$$c\vec{v} = \langle c v_1, c v_2 \rangle$$

$$\vec{v} - \vec{u} = \langle v_1 - u_1, v_2 - u_2 \rangle$$

3d $\vec{v} = \langle v_1, v_2, v_3 \rangle$ $\vec{u} = \langle u_1, u_2, u_3 \rangle$

$$\vec{v} + \vec{u} = \langle v_1 + u_1, v_2 + u_2, v_3 + u_3 \rangle$$

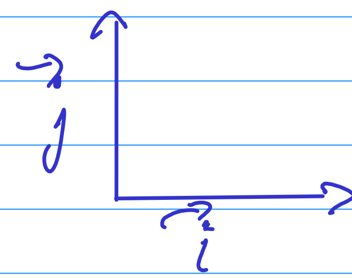
$$c\vec{v} = \langle cv_1, cv_2, cv_3 \rangle$$

$$\vec{v} - \vec{u} = \langle v_1 - u_1, v_2 - u_2, v_3 - u_3 \rangle$$

There is another way to write vectors, as combinations of basis vectors

In 2d $\vec{i} = \langle 1, 0 \rangle$

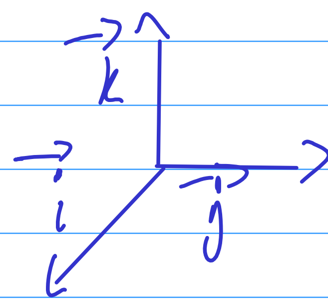
$\vec{j} = \langle 0, 1 \rangle$



In 3d $\vec{i} = \langle 1, 0, 0 \rangle$

$\vec{j} = \langle 0, 1, 0 \rangle$

$\vec{k} = \langle 0, 0, 1 \rangle$



Any vector can be written in terms of i, j, k

$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$

$$= \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle$$

$$= a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle$$

$$= a_1 \, i + a_2 \, j + a_3 k.$$

So components appear as coefficients of basis vectors.

In 2d $\vec{a} = \langle a_1, a_2 \rangle = a_1 \vec{i} + a_2 \vec{j}$

A vector $\vec{v}$ is a unit vector
if it has magnitude 1:

$$\vec{v} \text{ is unit vector}$$
$$\Updownarrow$$
$$\|\vec{v}\| = 1$$

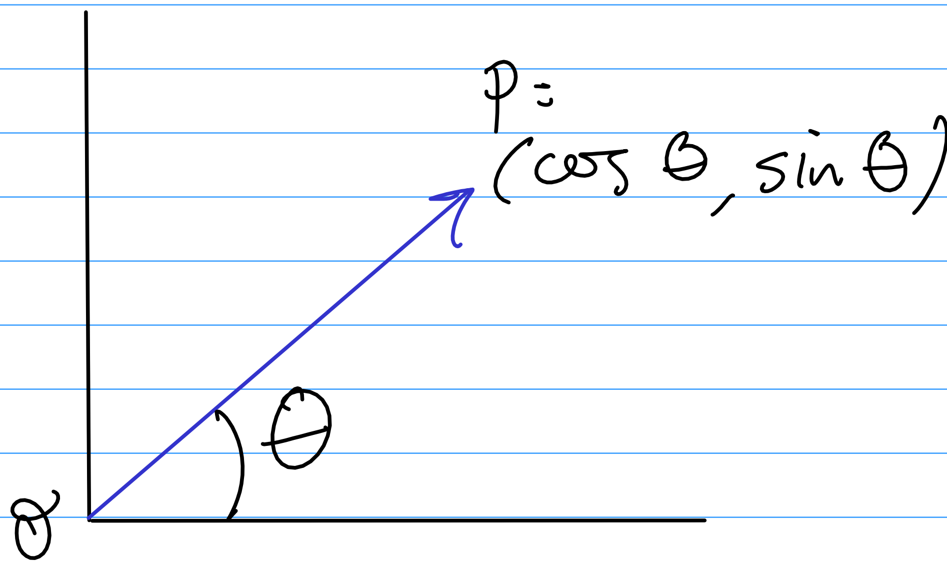
* $\vec{i}$, $\vec{j}$, and $\vec{k}$ are unit vectors.

* For any nonzero vector $\vec{v}$,
there is a unit vector with the
same direction as $\vec{v}$,

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{\vec{v}}{\|\vec{v}\|}.$$

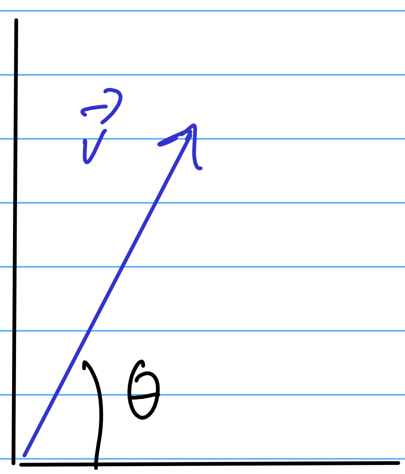
Since it's a scalar multiple, it has the
same direction, and $\left\|\frac{\vec{v}}{\|v\|}\right\| = \frac{1}{\|v\|}\|\vec{v}\| = 1.$

The unit vector in the direction that makes an angle θ with the horizontal



is $\vec{OP} = \langle \cos\theta, \sin\theta \rangle$

$= \cos\theta\, \vec{i} + \sin\theta\, \vec{j}$



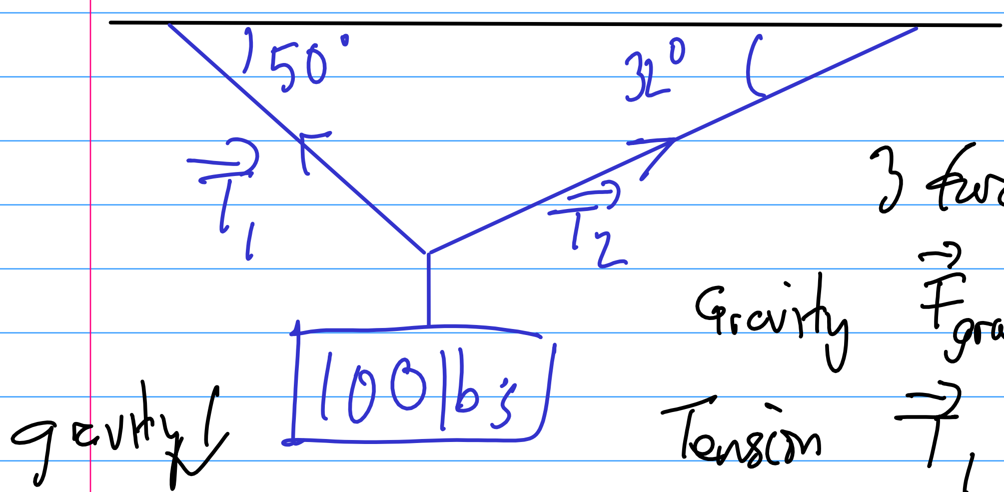
If $\vec{v}$ is any vector in this direction

$$\frac{\vec{v}}{\|\vec{v}\|} = \langle \cos\theta, \sin\theta \rangle$$

$$\vec{v} = \|\vec{v}\| \langle \cos\theta, \sin\theta \rangle$$

Application of vector arithmetic:
Forces and static equilibrium

In physics, a force is most naturally represented as a vector $\vec{F}$.
If an object is subject to several forces, the total or "resultant" force is the vector sum of the individual forces.



3 forces

Gravity $\vec{F}_{grav} = -100\vec{j}$

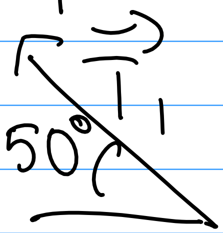
Tension $\vec{T}_1$ left side

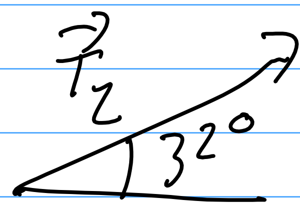
Tension $\vec{T}_2$ right side

In order for the object to be stationary, the forces must cancel:

$$\vec{T}_1 + \vec{T}_2 + \vec{F}_{grav} = 0$$

We can expand $\vec{T}_1$ and $\vec{T}_2$ into components





$$\vec{T}_1 = \|\vec{T}_1\| \langle -\cos 50^\circ, \sin 50^\circ \rangle$$

$$= -\|\vec{T}_1\| \cos 50^\circ \, i + \|\vec{T}_1\| \sin 50^\circ \, j$$

$$\vec{T}_2 = \|\vec{T}_2\| \langle \cos 32^\circ, \sin 32^\circ \rangle$$

$$= \|\vec{T}_2\| \cos 32^\circ \, \vec{i} + \|\vec{T}_2\| \sin 32^\circ \, \vec{j}$$

So the equation

$\vec{T_1} + \vec{T_2} + F_{grav} = 0$ becomes

$$\left(-\|T_1\| \cos 50^\circ + \|T_2\| \cos 32^\circ\right) \vec{i}$$

$$+ \left(\|T_1\| \sin 50^\circ + \|T_2\| \sin 32^\circ - 100\right) \vec{j}$$

$= 0$

gives 2 component equations

$$-\|T_1\| \cos 50^\circ + \|T_2\| \cos 32^\circ = 0$$

$$\|T_1\| \sin 50^\circ + \|T_2\| \sin 32^\circ = 100$$

2 linear equations in 2 unknowns $\|T_1\|$ and $\|T_2\|$

$\Rightarrow$ can solve.