Vector-Valued functions

A vector-valued function is a function with input t, a real number, and output $\vec{r}(t)$, a vector.

(Today we will be working in 3 dimensions)

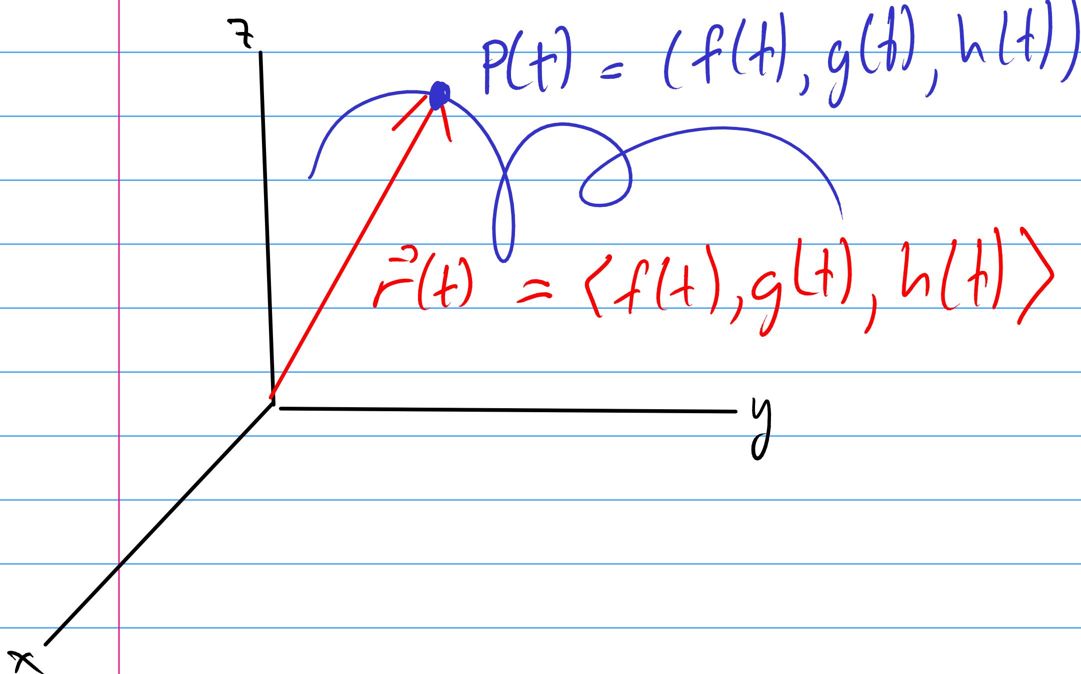
As usual the independent variable t may be called "the parameter" or "time"

If we write a vector function $\vec{r}(t)$ in components, we see that it consists of 3 real-valued functions:

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$
$$= f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$$

If we plot the points in 3-dimensional space whose position vectors are described by $\vec{r}(t)$, we get a parametric space curve

$$\vec{r}(t) = \vec{OP}(t)$$



<u>Example</u>: describe the space curve given by $\vec{r}(t) = \langle 1+t, 3t, -t \rangle$

$$\left\{ \begin{array}{l} x = 1+t \\ y = 3t \\ z = -t \end{array} \right\}$$

It's a line

$$\vec{r}_0 = \vec{r}(0) = \langle 1, 0, 0 \rangle$$

$$\vec{v} = \langle 1, 3, -1 \rangle$$

* So we were already dealing with vector-valued functions last week.

* Also, 2-dimensional parametric curves can be represented in the vector notation

$$\left\{ \begin{array}{l} x = f(t) \\ y = g(t) \end{array} \right\} \text{ becomes } \vec{r}(t) = f(t)\,\vec{i} + g(t)\,\vec{j}$$

where $\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$ in a 2-dimensional context.

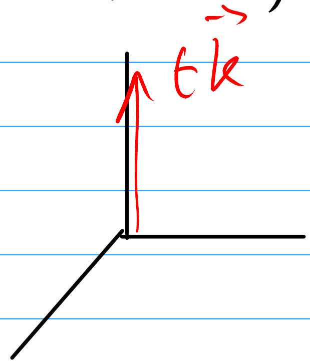
* For visualizing space curves, sometimes it is useful to forget some of the dimensions and see what we get. (Similar to the technique of using traces to visualize surfaces in 3d).

Ex Draw the curve

$$\vec{r}(t) = \underbrace{\cos t\,\vec{i} + \sin t\,\vec{j}}\, + \underbrace{t\vec{k}}$$

Idea: group the terms this way

The last term $t\vec{k}$ is something we've seen before: it describes a line, namely the z-axis.

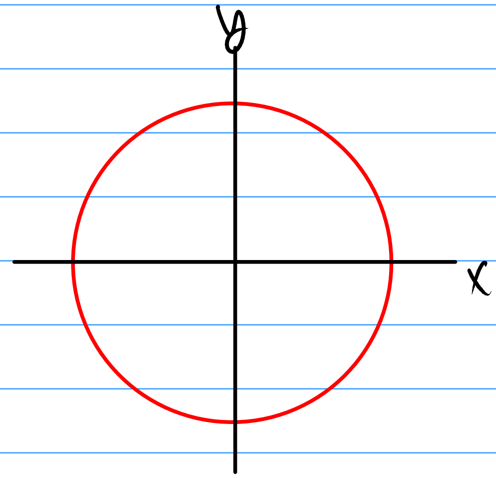


The first two terms are also something we've seen before:

$$\cos t\,\vec{i} + \sin t\,\vec{j} = \begin{cases} x = \cos t \\ y = \sin t \end{cases}$$

This is uniform circular motion

centered at $x=0$, $y=0$.



with radius 1

So we have circular motion in the x and y coordinates, combined with linear motion in the z coordinate.

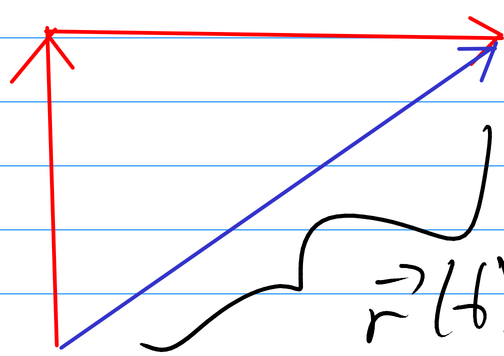
Now we have to combine them:

Recall vector addition:

$\cos t\,\vec{i} + \sin t\,\vec{j}$ circular motion

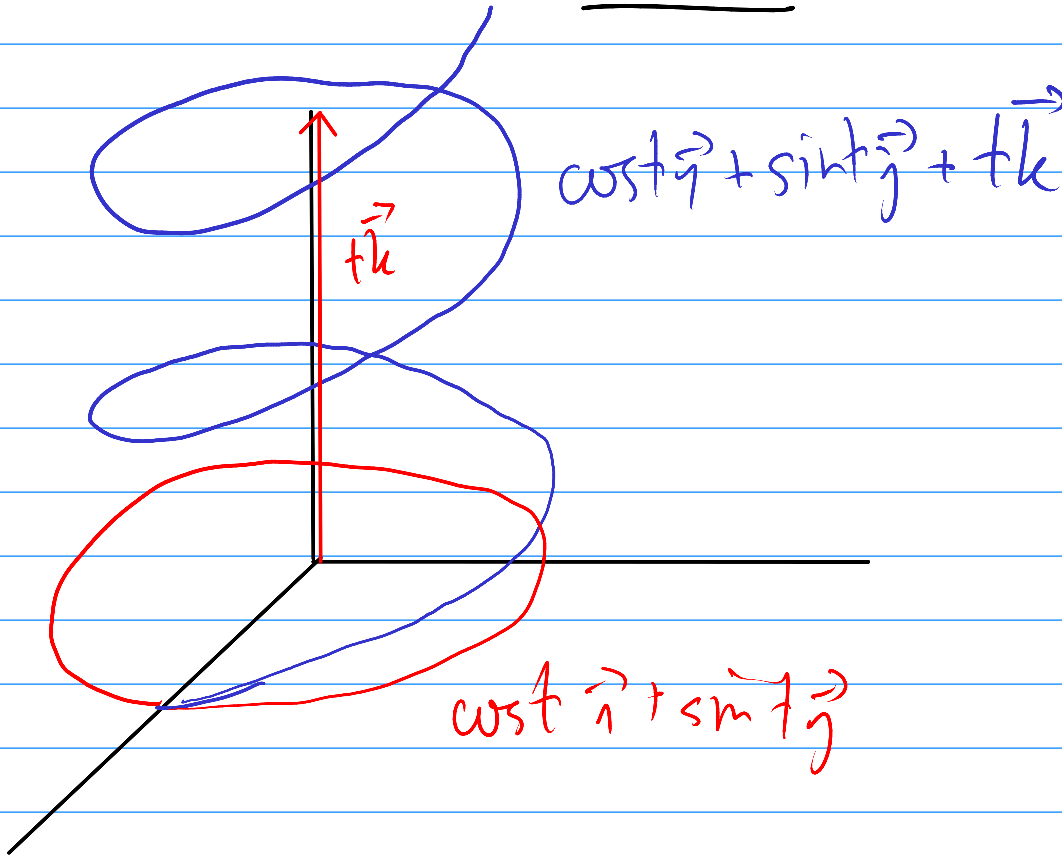
$t\vec{k}$

linear motion



$\vec{r}(t)$ total motion.

The result is a <u>helix</u>



This is the shape that appears in the structure of the DNA molecule, a double helix.

Sometimes we use multiple views of the same curve:

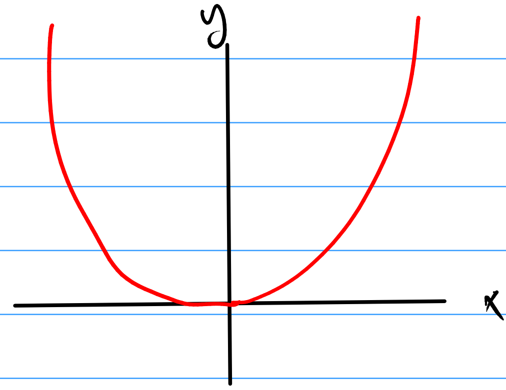
$$\vec{r}(t) = \langle t, t^2, t^3 \rangle = t\vec{i} + t^2\vec{j} + t^3\vec{k}$$

called a twisted cubic curve.

Look at xy-projection:

$$t\vec{i} + t^2\vec{j} \quad \begin{Bmatrix} x = t \\ y = t^2 \end{Bmatrix} \to y = x^2$$

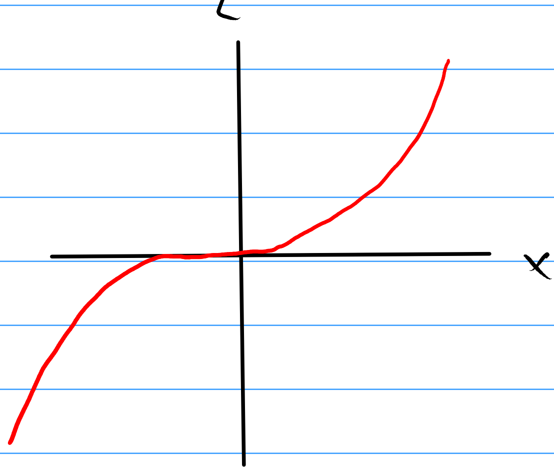
PARABOLA



Look at xz-projection

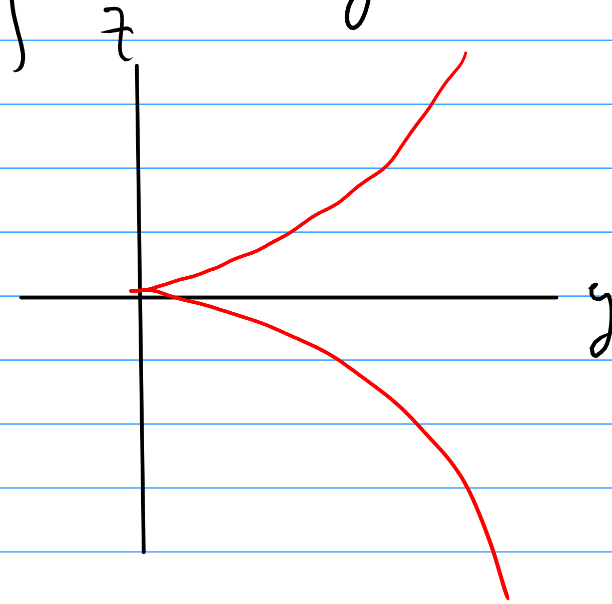
$$t\vec{i} + t^3\vec{k} \quad \begin{Bmatrix} x = t \\ z = t^3 \end{Bmatrix} \to z = x^3$$

CUBIC POLYNOMIAL



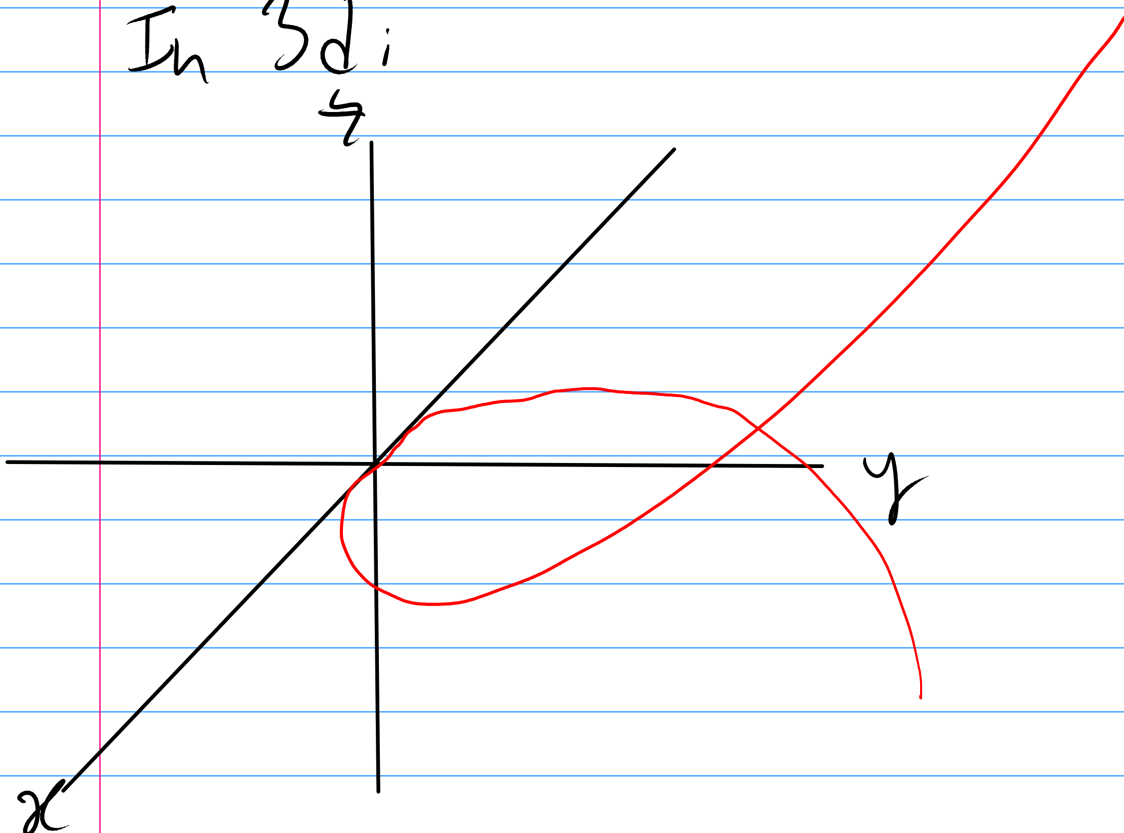
look at yz-projection

$t^2\vec{j} + t^3\vec{k}$ $\quad \begin{cases} y = t^2 \\ z = t^3 \end{cases}$ $\quad z = \pm y^{3/2}$



CUSP

In 3d:

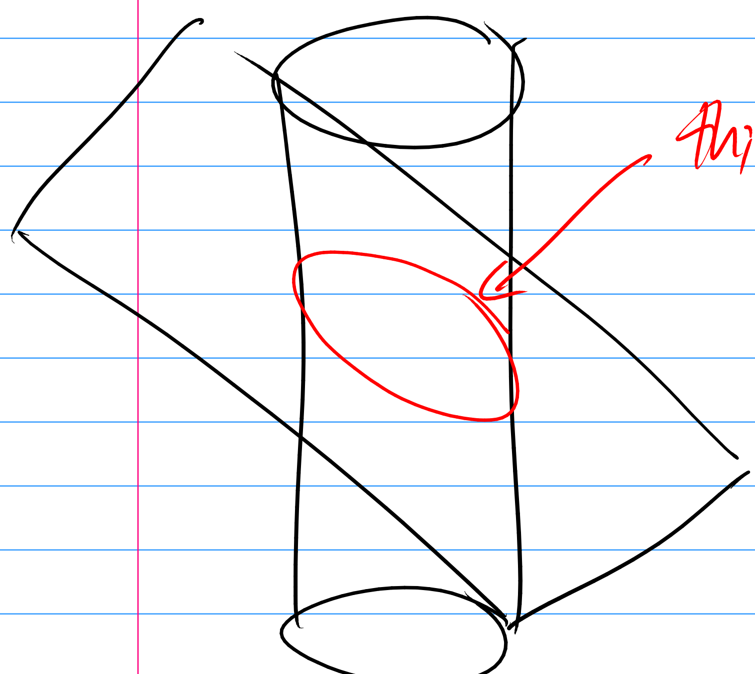


You can sometimes use this trick to construct a parameterization of a curve.

Ex Parameterize the intersection of the curves $x^2+y^2=1$ and $2x+z=1$.

$x^2+y^2=1$ cylinder over a circle

$2x+z=1$ plane normal vector $\langle 2,0,1\rangle$.



Because $x^2+y^2=1$, we know the x and y components will move in a circle

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad 0 \le t \le 2\pi$$

Now the other equation gives us z:
$$z = 1 - 2x = 1 - 2\cos t$$

So we get $\vec{r}(t) = \langle \cos t, \sin t, 1 - 2\cos t \rangle$
$$= \cos t\, \vec{i} + \sin t\, \vec{j} + (1 - 2\cos t)\, \vec{k}$$

Basic calculus for vector-valued functions

limits, derivatives, and integrals can be done component by component:

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$
$$= f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

$$\lim_{t \to a} \vec{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

$$\frac{d}{dt} \vec{r}(t) = \vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$$
$$= f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}$$

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle$$

$$= \left(\int_a^b f(t)\right)\vec{i} + \left(\int_a^b g(t)\,dt\right)\vec{j} + \left(\int_a^b h(t)\,dt\right)\vec{k}$$

Vector-valued fundamental theorem:

$$\int_a^b \frac{d}{dt}[\vec{r}(t)]\,dt = \vec{r}(b) - \vec{r}(a)$$

$$\frac{d}{dt}\int_a^t \vec{r}(t')\,dt' = \vec{r}(t)$$

Reason: the basis vectors $\vec{i}, \vec{j}, \vec{k}$ are constant in t, so they just come out like any other constant.

$$\frac{d}{dx}\left[(\text{constant})\cdot f(x)\right] = (\text{constant})\cdot \frac{d}{dx} f(x)$$

$$\int (\text{constant})\cdot f(x)\,dx = (\text{constant})\cdot \int f(x)\,dx$$

Using this basic calculus, we can formulate kinematics, the mathematical description of physical motion;

Position of a particle $\iff$ vector-valued function $\vec{r}(t)$

Velocity $\iff$ first derivative $\vec{v}(t) = \frac{d}{dt}\vec{r}(t) = \vec{r}\,'(t)$

acceleration $\iff$ second derivative $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) = \frac{d^2}{dt^2}\vec{r}(t)$

Newton's Second Law: $\vec{F} = m\vec{a}$

where $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

and $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

are vectors.

Basic example

Motion of a particle subject to a constant force. $\vec{F}$

Initial position: $\vec{r}_0 = \vec{r}(0)$

Initial velocity: $\vec{v}_0 = \vec{v}(0) = \vec{r}\,'(0)$

acceleratn: $\vec{a}(t) = \frac{\vec{F}}{m}$ constant.

$$\vec{v}\,'(t) = \vec{a}(t) = \frac{\vec{F}}{m}$$

$$\vec{v}(t) - \vec{v}_0 = \int_0^t \vec{v}\,'(s)\,ds$$ (s is a dummy variable)

$$= \int_0^t \frac{\vec{F}}{m}\,ds = \frac{\vec{F}}{m}\int_0^t ds = \frac{\vec{F}}{m}(t-0) = \frac{\vec{F}}{m}t$$

$$\vec{v}(t) = \vec{v}_0 + \frac{\vec{F}}{m}t$$

Integrate again:

$$\vec{r}(t) - \vec{r}_0 = \int_0^t \vec{r}\,'(s)\,ds = \int_0^t \vec{v}(s)\,ds$$

$$= \int_0^t \left(\vec{v}_0 + \frac{\vec{F}}{m} s\right) ds = \vec{v}_0 \int_0^t ds + \frac{\vec{F}}{m} \int_0^t s\,ds$$

$$= \vec{v}_0 (t - 0) + \frac{\vec{F}}{m}\left(\frac{1}{2}t^2 - \frac{1}{2}0^2\right)$$

$$= \vec{v}_0 t + \frac{\vec{F}}{m}\frac{1}{2}t^2$$

so $\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\frac{\vec{F}}{m} t^2$