Triple integrals

Bennett Contest

Sunday Dec. 4 2011 2:30-4:30 pm

RLM 6.124

Calculus level competition

(408D, 408M, 427L)

CASH PRIZES $$$

Sample question

Does $\sum_{n=1}^{\infty} \frac{1}{\ln(n!)}$ converge?

Triple Integrals

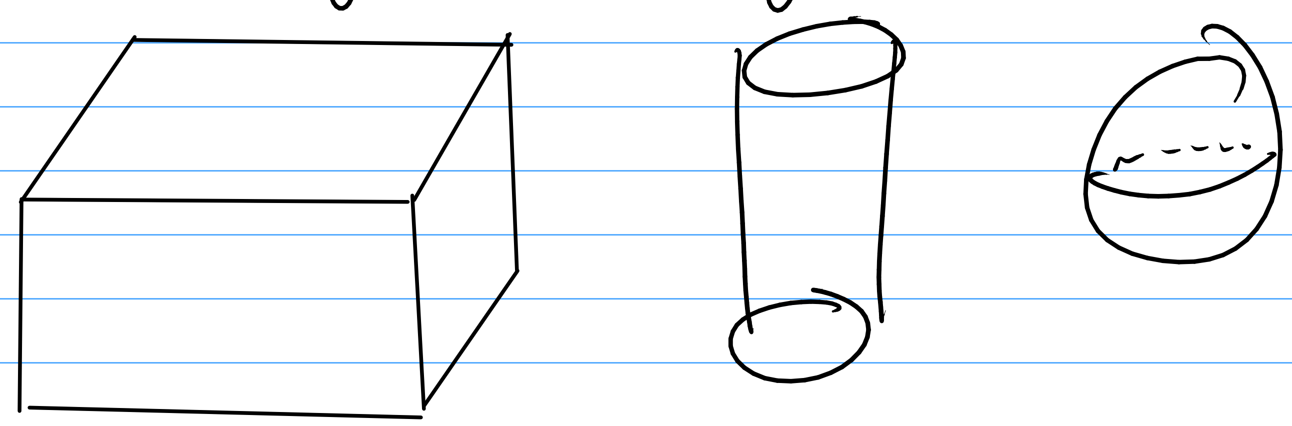
Double → Triple is just like single → double

$f(x,y,z)$

The domain is now a solid region in 3-dimensional space

Possible domains:

E: rectangular box Cylinder sphere



Want to define $\iiint_E f(x,y,z)\, \underbrace{dV}$

3-dimensional volume element (analogous to dA)

This integral does not necessarily represent a volume

Applications: mass vs density

density $= \dfrac{\text{mass}}{\text{volume}}$ constant density

E

dV

$\rho(x,y,z)$ variable density

$$dm = \rho(x,y,z)\,dV$$

$$m = \iiint_E dm = \iiint_E \rho(x,y,z)\,dV$$

total mass

Center of mass

$$M_x = \iiint_E x\rho\,dV$$

$$M_y = \iiint_E y\rho\,dV$$

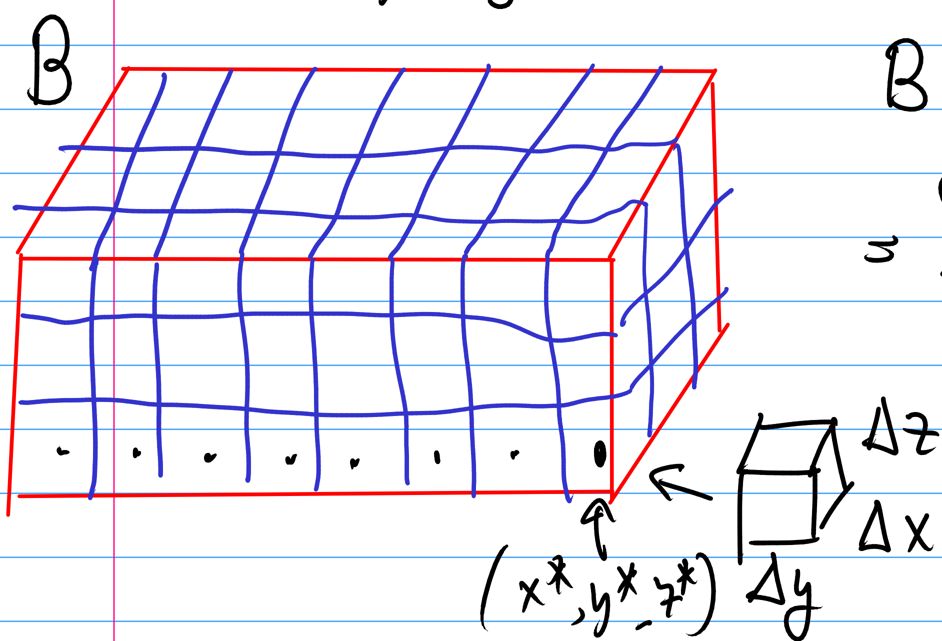
$$M_z = \iiint_E z\rho\,dV$$

$$\text{C.O.M.} = \left(\bar{x} = \frac{M_x}{m},\ \bar{y} = \frac{M_y}{m},\ \bar{z} = \frac{M_z}{m}\right)$$

$$\iiint_E 1\, dV = \text{Volume of } E$$

Definition by Riemann sums for a rectangular box

B



(x^*, y^*, z^*)

$$B = [a,b] \times [c,d] \times [r,s]$$

$$= \left\{ \begin{array}{c} a \le x \le b \\ c \le y \le d \\ r \le z \le s \end{array} \right\}$$

$$\Delta x = \frac{b-a}{l}$$

$$\Delta y = \frac{d-c}{m}$$

$$\Delta z = \frac{s-r}{n}$$

$$\Delta V = \Delta x \Delta y \Delta z$$

Sample from each small box

$$f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$$

Riemann Sum

$$= \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V$$

This is an approximation to the $\iiint_B f\,dV$

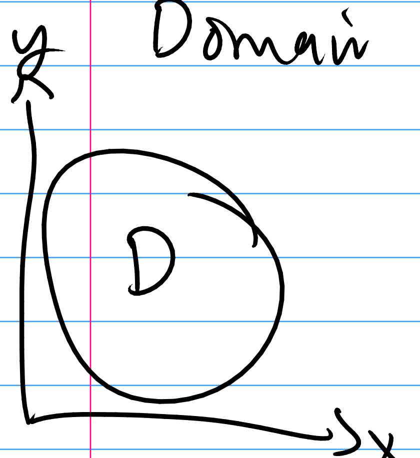
Take $\lim\limits_{l,m,n\to\infty}$ gives $\iiint_B f\,dV$

<u>Iterated integrals</u> $\quad B=[a,b]\times[c,d]\times[r,s]$

$$\iiint_B f\,dV = \int_r^s \int_c^d \int_a^b f \; dx\,dy\,dz$$

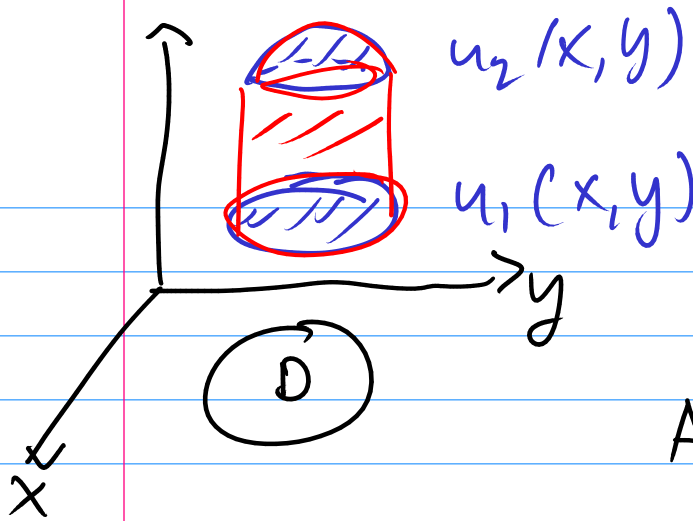
or $dx\,dz\,dy$ or $dy\,dx\,dz$
or $dy\,dz\,dx$ or $dz\,dx\,dy$
or $dz\,dy\,dx$)

Domain D in the xy-plane



$u_1(x,y)$ $\quad u_2(x,y)$ functions

$$E = \left\{(x,y,z) \;\middle|\; \begin{array}{l}(x,y) \text{ in } D \\ u_1(x,y)\le z\le u_2(x,y)\end{array}\right\}$$



E is the solid object between graphs of u_1 and u_2.

And we also have a function $f(x,y,z)$

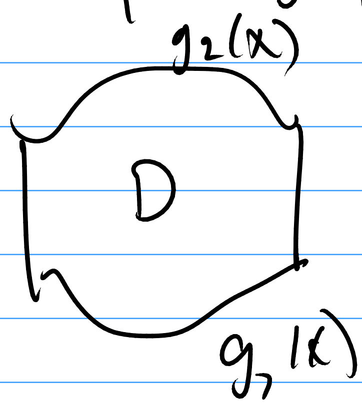
$$\iiint_E f(x,y,z)\,dV$$

integrate w.r.t. z first

$$= \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA$$

Suppose D is a Type I region

$$D = \{ (x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x) \}$$

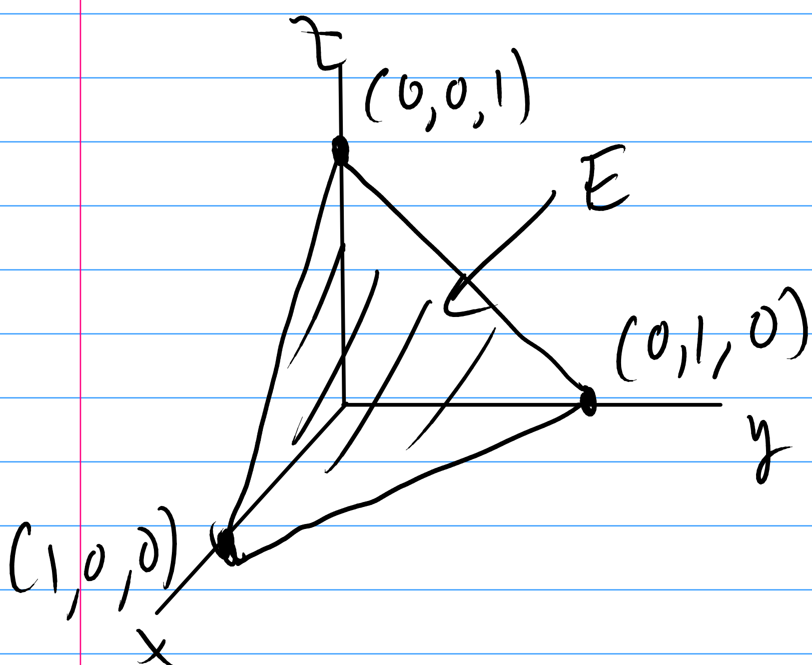


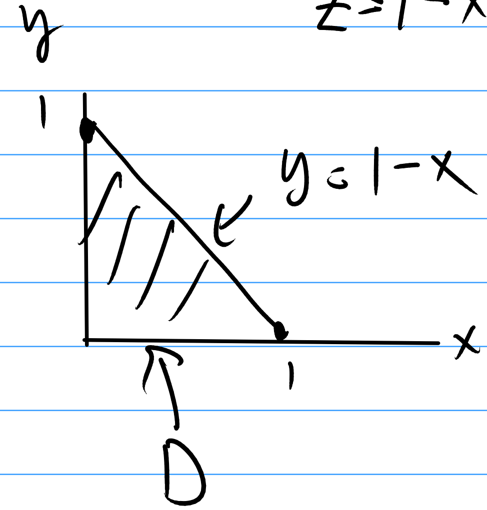
$$\iint_D [\ ]\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} [\ ]\,dy\,dx$$

$$\iiint_E f\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

Ex $\displaystyle\iiint_E z\,dV$ E = tetrahedron bounded by $x=0$, $y=0$, $z=0$, $x+y+z=1$

$z = 1-x-y$





$$E = \{(x,y,z) \mid (x,y)\in D,\ 0 \le z \le 1-x-y\}$$

$$D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1-x\}$$

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx$$

$$\int_0^1 \int_0^{1-x} \left[\int_0^{1-x-y} z \, dz \right] dy \, dx$$

$$\downarrow$$

$$\left[\frac{1}{2} z^2 \right]_0^{1-x-y} = \frac{1}{2} (1-x-y)^2$$

$$\int_0^{1-x} \frac{1}{2} (1-x-y)^2 \, dy$$

$$\left[-\frac{1}{6} (1-x-y)^3 \right]_{y=0}^{y=1-x}$$

$$= -\frac{1}{6} \left[0 - (1-x)^3 \right] = \frac{1}{6} (1-x)^3$$

$$\int_0^1 \frac{1}{6} (1-x)^3 \, dx = \left[-\frac{1}{24} (1-x)^4 \right]_0^1$$

$$= \frac{1}{24}$$

Other orders of integration

$$\int_r^s \int_{g_1(z)}^{g_2(z)} \int_{h_1(x,z)}^{h_2(x,z)} f(x,y,z)\, dy\, dx\, dz$$

$$E = \left\{ (x,y,z) \,\middle|\, \begin{array}{l} r \le z \le s \\ g_1(z) \le x \le g_2(z) \\ h_1(x,z) \le y \le h_2(x,z) \end{array} \right\}$$

Ex $\iiint_E yz\cos(x^5)\, dV$

$$E = \{ 0 \le x \le 1,\ 0 \le y \le x,\ x \le z \le 2x \}$$

$$\int_0^1 \int_0^x \int_x^{2x} yz\cos(x^5)\, dz\, dy\, dx$$

$$\int_0^1 \int_x^{2x} \int_0^x yz\cos(x^5)\, dy\, dz\, dx$$

$$= \int_0^1 \left[\frac{1}{2} x^2 \left(\frac{(2x)^2}{2} - \frac{x^2}{2} \right) \right] \cos(x^5)\, dx$$

$$= \int_0^1 \frac{3}{4} x^4 \cos(x^5)\, dx$$

$u = x^5 \quad du = 5x^4 dx$