Taylor Series

Last time used tricks to get power series representations

Today Systematic method for getting a power series from "any" function.

How: Work backwards

Suppose $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$

(0) Plug in $x=a$

$$f(a) = c_0 + \underbrace{c_1(a-a) + c_2(a-a)^2 + \cdots}_{0}$$

$$f(a) = c_0$$

(1) Take derivative and plug in $x=a$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

$$f'(a) = c_1$$

(2) $f''(x) = 2c_2 + 6c_3(x-a) + \cdots$

$f''(a) = 2c_2$

$c_2 = \dfrac{f''(a)}{2}$

(3) $f'''(x) = 3\cdot 2c_3 + 4\cdot 3 2c_4(x-a) + \cdots$

$f'''(a) = 3\cdot 2c_3 \qquad c_3 = \dfrac{f'''(a)}{3\cdot 2} = \dfrac{f'''(a)}{3!}$

$f^{(n)}(x) = n!\,c_n + (n+1)!\,c_{n+1}(x-a) + \cdots$

$f^{(n)}(a) = n!\,c_n$

$c_n = \dfrac{f^{(n)}(a)}{n!}$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$+ \frac{f'''(a)}{3!}(x-a)^3$$

Taylor series for f centered at a.

Taylor polynomials are the partial sums of the taylor series

$$\cancel{T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{n!}(x-a)^n}$$

$$T_N(x) = \sum_{n=0}^{n=N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor series centered at 0 for $f(x) = e^x$

$f(x) = e^x \qquad f(0) = 1$

$f'(x) = e^x \qquad f'(0) = 1$

$f''(x) = e^x \qquad f''(0) = 1$

$f'''(x) = e^x \qquad f'''(0) = 1$

Notice pattern: $f^{(n)}(0) = 1$ for all n.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

Radius of convergence $= \infty$

Taylor Series for $\sin x$ centered at 0.

$f(x) = \sin x$

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0$

Repeats ↓

$$f(x) = 0 + 1(x) + \frac{0\,x^2}{2!} - \frac{1\,x^3}{3!} + \frac{0x^4}{4!} + \frac{1x^5}{5!} - \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sin x$$

Radius of Convergence $= \infty$.

Taylor polynomial of <u>degree 3</u>

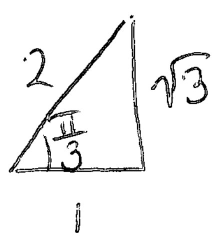
for $\cos x$ at $x = \frac{\pi}{3}$ $\quad \sum c_n\left(x - \frac{\pi}{3}\right)^n$

$f(x) = \cos x \qquad f\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$f'(x) = -\sin x \qquad f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$f''(x) = -\cos x \qquad f''\left(\frac{\pi}{3}\right) = -\frac{1}{2}$

$f'''(x) = \sin x \qquad f'''\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$



$$T_3(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{2}\left(\frac{1}{2!}\right)\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{2}\left(\frac{1}{3!}\right)\left(x - \frac{\pi}{3}\right)^3$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$T_0(x) = 0$$

$$T_1(x) = x$$

$$T_2(x) = x$$

$$T_3(x) = x - \frac{x^3}{3!}$$

$$T_4(x) = x - \frac{x^3}{3!} + \left(0\frac{x^4}{4!}\right)$$

$$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

Uses of power series: integrate functions term by term.

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$\int e^{-x^2}\,dx = \sum_{n=0}^{\infty} \int \frac{(-1)^n x^{2n}}{n!}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\,n!} + C$$

use instead of L'Hospital's Rule

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \to 0} \frac{1}{x^2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right) - 1 - x\right]$$

$$= \lim_{x \to 0} \frac{1}{x^2}\left[\frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right]$$

$$= \lim_{x \to 0} \left[\frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots\right]$$

$$= \frac{1}{2!} = \frac{1}{2}.$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right)}{x}$$

$$= \lim_{x \to 0} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \right)$$

$$= 1$$

Multiplication of Taylor series

$(a+b)(c+d) =$
$ac + ad + bc + bd$

$$e^x \sin x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$
$$\times \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

compute the product to order 3
(degree 3)

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\times \qquad x \quad - \frac{x^3}{3!} \quad + \frac{x^5}{5!}$$

$$x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots$$

$$- \frac{x^3}{3!} + \left(-\frac{x^4}{3!}\right) + \cdots$$

$$x + x^2 + \frac{1}{3}x^3 + \text{higher degree terms}$$