Taylor Series : Estimates & applications.

$$f(x) \rightarrow \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Can build a Taylor series for any differentiable function:

<u>Questions</u>: ① Convergence of Taylor Series?
what is its interval of convergence?

Ratio or Root Test → Radius of convergence.

② Does the Taylor series converge to <u>the original function</u>? How fast?

Estimate the error?

Notation $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ Taylor Series

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$ degree n Taylor polynomial.

Remainder $R_n(x) = f(x) - T_n(x)$

$R_n(x)$ is the error in approximating $f(x)$ by $T_n(x)$.

Convergence to f : $\displaystyle\lim_{n\to\infty} T_n(x) = f(x)$

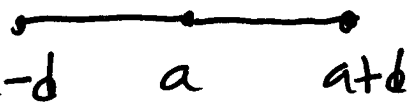
equivalently : $\displaystyle\lim_{n\to\infty} R_n(x) = 0$

If this is true, then $f(x)$ is the sum of the Taylor series at x.

Need To estimate $R_n(x)$

<u>Taylor's Theorem</u> (a = center of taylor series)

consider an interval of x such that $|x-a| \le d$



Suppose $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$.

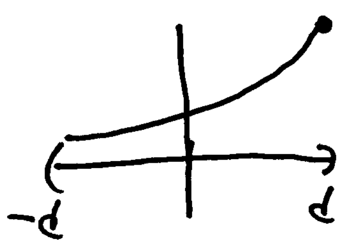
<u>Then</u>:

$$|R_n(x)| \le \frac{M}{(n+1)!} |x-a|^{n+1}$$

<u>Theorem</u> $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x.

<u>Proof</u> consider some interval $|x| \le d$

On this interval $f^{(n+1)}(x) = e^x$



$$|e^x| \le e^d = M$$

By Taylor's theorem

$$|R_n(x)| \le \frac{e^d}{(n+1)!} |x|^{n+1} \quad \text{for } |x| \le d$$

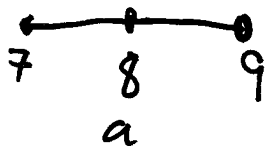
$$\lim_{n\to\infty} |R_n(x)| \le \lim_{n\to\infty} \frac{e^d}{(n+1)!} |x|^{n+1} = 0$$

(uses the fact $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for any x)

$$\lim_{n\to\infty} R_n(x) = 0 \Rightarrow e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Ex (a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.

(b) Estimate the accuracy of this approximation for x in the interval $7 \le x \le 9$.



(a)

$$\left.\begin{aligned} f(x) &= x^{1/3} & f(8) &= 2 \\ f'(x) &= \frac{1}{3}x^{-2/3} & f'(8) &= \frac{1}{12} \\ f''(x) &= \frac{-2}{9}x^{-5/3} & f''(8) &= -\frac{1}{144} \end{aligned}\right\} \text{enough to get degree 2 Taylor Polynomial}$$

$$f'''(x) = \frac{10}{27}x^{-8/3} \quad \Big\} \text{ To estimate the error}$$

$$T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$$

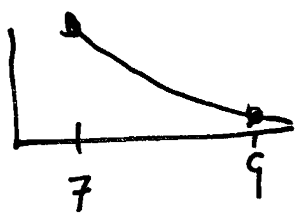
Part (a) ✓

Taylor's theorem: $R_2(x) = f(x) - T_2(x)$

$$|R_2(x)| \le \frac{M}{3!}|x-8|^3$$

where $M = \max\limits_{7 \le x \le 9} f'''(x) = \max\limits_{7 \le x \le 9} \frac{10}{27} x^{-8/3}$

Since $x^{-8/3}$ is decreasing, $M = \frac{10}{27}(7)^{-8/3}$



< 0.0021

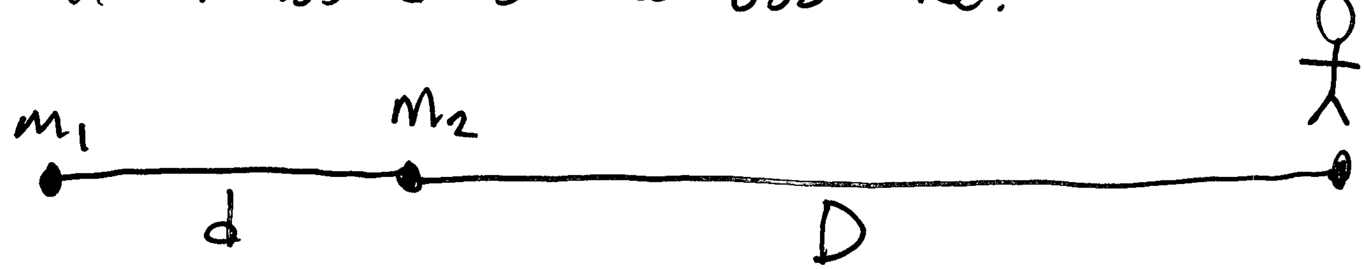
$$|R_2(x)| \le \frac{0.0021}{3!}|x-8|^3$$

Furthermore if $7 \le x \le 9$, then $|x-8| \le 1$

$$|R_2(x)| \le \frac{0.0021}{3!} 1^3 < 0.0004$$

A physics problem

Two massive planets, masses m_1 & m_2 a distance d apart. An observer is located a distance D away ($D \gg d$) (along the line joining the planets) what mass does he observe?



Gravitational Field Strength

$$= \frac{GM}{R^2}$$

G = Newton's Constant
M = mass of planet
R = distance from planet to observer.

Field strength in our case

$$F = \frac{Gm_1}{(D+d)^2} + \frac{Gm_2}{D^2} \quad \text{vs.} \quad \frac{G(m_1+m_2)}{D^2}$$

If d is much smaller than D,

then $\frac{d}{D} \ll 1$ is very small,

and we can take the taylor series for our function using $\frac{d}{D}$ as the variable.

$$\frac{1}{(D+d)^2} = \frac{1}{D^2\left(1+\frac{d}{D}\right)^2}$$

$$\boxed{\alpha=-2} \quad \frac{1}{\left(1+\frac{d}{D}\right)^2} = \left(1 - 2\frac{d}{D} + 3\left(\frac{d}{D}\right)^2 + \cdots\right)$$

Binomial series

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

For Real α.

$$\frac{GM_1}{(D+d)^2} + \frac{GM_2}{D^2} = \frac{GM_1}{D^2}\left(\frac{1}{(1+\frac{d}{D})^2}\right) + \frac{GM_2}{D^2}$$

$$= \frac{GM_1}{D^2}\left(\underline{\underline{\underline{1}}} - 2\frac{d}{D} + 3\left(\frac{d}{D}\right)^2 + \cdots\right) + \frac{GM_2}{D^2}$$

just take the first term of the taylor series

$$\approx \frac{GM_1}{D^2} + \frac{GM_2}{D^2} = \frac{G(M_1+M_2)}{D^2}$$