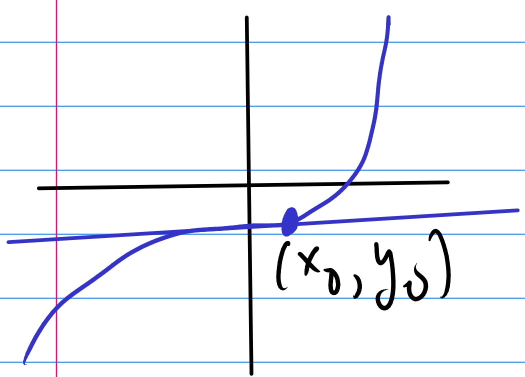
Tangent Planes / Linear Approximation

Recall Single Variable calculus

$f'(x)$ is the slope of the tangent line



equation of tangent line

$(y - y_0) = f'(x_0)(x - x_0)$

$[\, y_0 = f(x_0) \,]$

Linear approximation

$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$

differential: $dy = f'(x)\,dx$

2-variables $f(x,y)$

suppose (x_0, y_0, z_0) is on the graph of $z = f(x,y)$

A plane passing through this point

has an equation of the form

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0$$

rearrange & divide by $-C$

$$z-z_0 = a(x-x_0)+b(y-y_0)$$

$$\left(a = -\frac{A}{C},\ b = -\frac{B}{C}\right)$$

a is the slope when y is held constant

$$a = f_x(x_0,y_0) = \frac{\partial f}{\partial x}(x_0,y_0)$$

b is the slope when x is held constant

$$b = f_y(x_0,y_0) = \frac{\partial f}{\partial y}(x_0,y_0)$$

Equation for the tangent plane

$$z-z_0 = f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

$z = f(x,y) = 2x^2 + y^2$ at $(1,1,3)$

$$f_x(x,y) = \frac{\partial}{\partial x}(2x^2+y^2) = \frac{\partial}{\partial x}(2x^2) + \frac{\partial}{\partial x}(y^2)$$

$$= 2\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(y^2)$$

$$= 2(2x) + 0 = 4x$$

$$f_y(x,y) = \frac{\partial}{\partial y}(2x^2+y^2) = \frac{\partial}{\partial y}(2x^2) + \frac{\partial}{\partial y}(y^2)$$

$$= 0 + 2y = 2y$$

At $(1,1,3)$ $\quad f_x(1,1) = 4 \quad f_y(1,1) = 2$

$$z - z_0 = 4(x - x_0) + 2(y - y_0)$$

is the equation of the tangent plane.

Linear approximation

Replace graph by the tangent plane, and approximate the function by a linear function (the linearization of f)

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

linearization

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$f(x,y) \approx L(x,y)$$

use this to approximate

$f(x,y) = 2x^2+y^2$ $\qquad f(1,1) = 3$

$f(1,1,\ 0.95)$ $\qquad f_x(1,1) = 4$

$f_y(1,1) = 2$

$$L(x,y) = 3 + 4(x-1) + 2(y-1)$$

$$L(1.1, 0.95) = 3 + 4(0.1) + 2(-0.05)$$

$$= 3 + 0.4 - 0.1 = 3.3$$

$$f(1.1, 0.95) \approx 3.3$$

Differentiability means that tangent plane / linear approx. works well

$\Delta z = z - z_0$ $\quad \Delta x = x - x_0$ $\quad \Delta y = y - y_0$

Definition f is differentiable if

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as Δx and $\Delta y \to 0$

Theorem:

If the partial derivatives f_x and f_y exist and are continuous, then f is differentiable in this sense.

Differentials are a convenient notation for linear approximation $z = f(x,y)$

$dx = \Delta x$ $\qquad$ $dy = \Delta y$

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x, y)$$

$dz =$ linear approximation to Δz

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Easy to remember

$z = f(x,y) = 2x^2 + y^2$ $\qquad$ $(1,1) \to (1.1, 0.95)$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$\frac{\partial z}{\partial x} = 4x$$

$$\frac{\partial z}{\partial y} = 2y$$

$$dz = 4x\,dx + 2y\,dy$$

plug in $x = 1$, $y = 1$, $dx = 0.1$, $dy = -0.05$

$$dz = 4(0.1) - 2(0.05) = 0.3$$

$$f(1, 1, 0.95) \approx f(1,1) + dz$$
$$3 + 0.3 = 3.3$$

<u>3 variables</u> $w = f(x,y,z)$

$$f_x = \frac{\partial w}{\partial x} \qquad f_y = \frac{\partial w}{\partial y} \qquad f_z = \frac{\partial w}{\partial z}$$

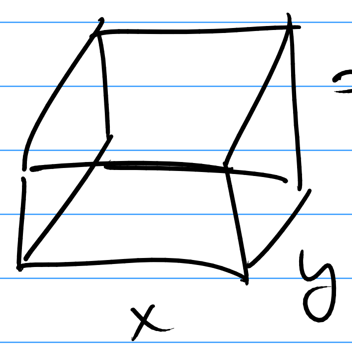
$$\begin{aligned} w - w_0 &= f_x(x_0, y_0, z_0)(x - x_0) \\ &+ f_y(x_0, y_0, z_0)(y - y_0) \\ &+ f_z(x_0, y_0, z_0)(z - z_0) \end{aligned}$$

linear approx near (x_0, y_0, z_0, w_0)

where $w_0 = f(x_0, y_0, z_0)$

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

Ex



Volume $= V = xyz$

initially sides are $\overset{x}{10}, \overset{y}{20}, \overset{z}{30}$ $V = 6000$

Estimate V if sides increase by 1

$$dV = (yz)dx + (xz)dy + (xy)dz$$

$$20 \cdot 30 \cdot 1 + 10 \cdot 30 \cdot 1 + 10 \cdot 20 \cdot 1$$

$$600 + 300 + 200 = 1100$$

$$V + dV = 7100$$

true value $= 11 \cdot 21 \cdot 31 = 7161$

error $= 61$ which is 0.8% of the true value. Quite good.