Strategy for Testing Series

1) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$

   diverges if $p \leq 1$

2) Geometric Series $\sum ar^{n-1}$ or $\sum ar^n$

   converges if $|r| < 1$

   diverges if $|r| \geq 1$

3) Comparison tests

   Consider series $\sum a_n$, $\sum b_n$ $a_n, b_n \geq 0$

   if $a_n \leq b_n$ if $\sum b_n$ converges

   then $\sum a_n$ converges

   if $b_n \leq a_n$ if $\sum b_n$ diverges

   then $\sum a_n$ diverges

   if $\lim \frac{a_n}{b_n} = c$ is finite and > 0

   then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

Use if: $\sum a_n$ is similar to p-series or geometric series.

Ex algebraic functions $\dfrac{\sqrt{n^5+2}}{n^6+3n^2}$ vs. $\dfrac{\sqrt{n^5}}{n^6} = n^{-7/2}$

if a_n is not necessarily positive, can look at $\sum |a_n|$, and try for absolute convergence.

4. Remember to check if $\lim_{n\to\infty} a_n = 0$.

Test for divergence: if $\lim_{n\to\infty} a_n \neq 0$, $\sum a_n$ diverges.

5. $\sum (-1)^{n-1} b_n$ or $\sum (-1)^n b_n$, $b_n \geq 0$.

* Need $b_{n+1} \leq b_n$ (decreasing)

* $\lim_{n\to\infty} b_n = 0$.

Then Alternating series test $\Rightarrow$ convergence

6. any series $\sum a_n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$$

if $L<1 \Rightarrow \sum a_n$ converges (absolutely)

if $L>1 \Rightarrow \sum a_n$ diverges

if $L=1 \Rightarrow$ ~~[illegible]~~ inconclusive

$\frac{a_{n+1}}{a_n}$ simplifies $\begin{cases} \text{factorials } n! \\ 2^n \end{cases}$

$$\frac{2^{n+1}}{2^n} = 2 \qquad \frac{(n+1)!}{n!} = n+1$$

always inconclusive if a_n is an algebraic function of n.

$$a_n = n^3 \Rightarrow \frac{a_{n+1}}{a_n} = \frac{(n+1)^3}{n^3} = \left(1+\frac{1}{n}\right)^3 \to 1$$

7. if $a_n = (b_n)^n$ use root test

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$$

if $L < 1 \Rightarrow$ converges (absolutely)

if $L > 1 \Rightarrow$ diverges

if $L = 1 \Rightarrow$ inconclusive.

If ratio test is inconclusive, so is the root test and vice versa.

8. if $a_n = f(n)$ and $\int_1^\infty f(x)\,dx$ seems reasonably easy, use integral test

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^\infty f(x)\,dx = \lim_{t\to\infty} \int_1^t f(x)\,dx$$

either both converge or both diverge.

Ex $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$ Alternating

Need decreasing and lim = 0.

$$\lim_{n\to\infty} \frac{n}{n+2} = 1$$ Test for divergence says it diverges. the series

$$= \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}}$$

Ex $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+2}$ Alternating series

$$\lim_{n\to\infty} \frac{n}{n^2+2} = \lim_{n\to\infty} \frac{1}{n+\frac{2}{n}} = 0$$

decreasing? ~~Study~~ Need $b_{n+1} \leq b_n$

$$\boxed{\frac{(n+1)}{(n+1)^2+2} \leq \frac{n}{n^2+2}} \ ?$$

Study $f(x) = \frac{x}{x^2+2}$. Look at $f'(x) = \frac{(x^2+2) - x(2x)}{(x^2+2)^2}$

$$= \frac{2-x^2}{(x^2+2)^2} < 0.$$

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+3^n}$ compare with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$

$\dfrac{1}{n+3^n} < \dfrac{1}{3^n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ is geometric with $r = \dfrac{1}{3}$

$\Rightarrow$ converges

by comparison test, $\displaystyle\sum \frac{1}{n+3^n}$ converges

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}}$ Root test

$$\lim_{n\to\infty} \sqrt[n]{\frac{(2n+1)^n}{n^{2n}}} = \lim_{n\to\infty} \frac{2n+1}{n^2} = \lim_{n\to\infty} \frac{2}{n} + \frac{1}{n^2} = 0$$

$0 < 1 \Rightarrow$ converges.

So $f'(x) < 0$ if $x > \sqrt{2}$.

Good enough for alternating series test.

So $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+2}$ converges.

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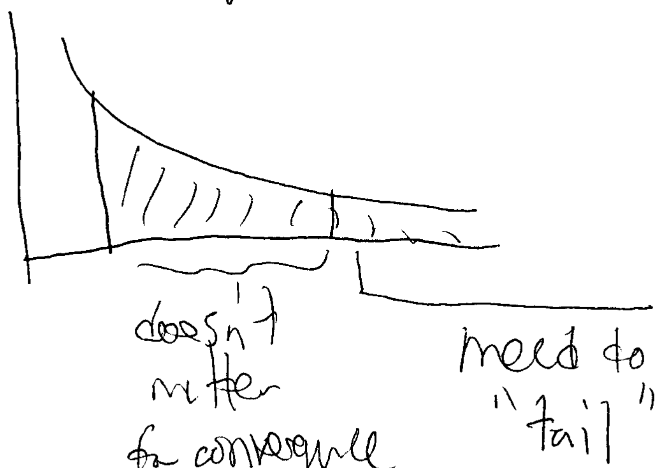
for improper integrals



for series

$$\sum_{n=1}^{\infty} a_n = \underbrace{\sum_{n=1}^{1000000} a_n}_{\text{doesn't matter for convergence}} + \underbrace{\sum_{n=1000001}^{\infty} a_n}_{\text{need to know if the "tail" converges.}}$$