Series

In Calculus, the term "series" or "infinite series" is used denote what happens when we try to add together the elements of a sequence.

sequence $\quad a_1, a_2, a_3, \dots \quad \{a_n\}_{n=1}^{\infty}$

series $\quad a_1 + a_2 + a_3 + \dots = \sum_{n=1}^{\infty} a_n$

Q: How is it possible to add together infinitely many numbers and get a finite answer?

A: It isn't always possible: the situation is similar to the ~~[illegible]~~ improper integral $\int_1^{\infty} f(x)\,dx$; there is always a question of convergence.

Improper integral $\int_1^\infty f(x)\,dx$

⇝ function $\int_1^t f(x)\,dx = F(t)$

⇝ limit $\lim_{t\to\infty} F(t) = \lim_{t\to\infty} \int_1^t f(x)\,dx$

If this limit converges to a finite limit, that number is $= \int_1^\infty f(x)\,dx$, by definition

Infinite series $\sum_{n=1}^\infty a_n$

⇝ $\boxed{\text{sequence of partial sums}}$ $S_n = \sum_{i=1}^n a_i$

= sum of first n terms of the series.

⇝ limit $\lim_{n\to\infty} S_n$

$= \lim_{n\to\infty} \sum_{i=1}^n a_i$

If this limit converges to a finite value, that number is $= \sum_{n=1}^\infty a_n$, by definition.

Ex $a_i = i$ original sequence, $\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} i$ series

$s_1 = a_1 = 1$

$s_2 = a_1 + a_2 = 1 + 2 = 3$

$s_3 = a_1 + a_2 + a_3 = 1 + 2 + 3 = 6$

$s_4 = a_1 + a_2 + a_3 + a_4 = 1 + 2 + 3 + 4 = 10$

One can show $s_n = \frac{n(n+1)}{2}$ sequence of partial sums

Hence $\lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{n(n+1)}{2} = \infty$

The series diverges

Ex $a_i = \frac{1}{2^i}$ sequence, $\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} \frac{1}{2^i}$ series

$s_1 = a_1 = \frac{1}{2}$

$s_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

One can show

$s_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$

Thus $\lim_{n \to \infty} s_n = \lim_{n \to \infty} 1 - \frac{1}{2^n} = 1$

The series $\sum_{i=1}^{\infty} \frac{1}{2^i}$ converges

and the sum is $\boxed{1}$.

$\sum_{i=1}^{\infty} \frac{1}{2^i}$ is an example of a <u>geometric series</u>

"geometric" means ratio of successive terms is independent of i : $\frac{a_{i+1}}{a_i} = r$ = "common ratio"

general form : $a_i = a r^{i-1}$

$= (\text{first term})(\text{common ratio})^{i-1}$

<u>Theorem</u> If $|r|<1$, the geometric series

$$\sum_{i=1}^{\infty} a r^{i-1} = a + ar + ar^2 + \cdots$$

converges, and $\sum_{i=1}^{\infty} ar^{i-1} = \frac{a}{1-r}$; $|r|<1$

If $|r| \geq 1$, the series diverges.

$$\sum_{i=1}^{\infty} (\text{first term})(\text{common ratio})^{i-1} = \frac{(\text{first term})}{1-(\text{common ratio})}$$

Proof $S_n = a + ar + \cdots + ar^{n-1}$

$rS_n = \quad ar + \cdots + ar^{n-1} + ar^n$

$S_n - rS_n = a - ar^n = a(1-r^n)$

$S_n = \dfrac{a(1-r^n)}{1-r}$ partial sum of the geometric series

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \dfrac{a(1-r^n)}{1-r} = \dfrac{a}{1-r}$

since $\lim_{n\to\infty} r^n = 0$ of $|r|<1$.

If $|r|>1$ or $r=-1$, limit doesn't exist

if $r=1$, get 0 in denominator :(

Ex $\quad 2 + \frac{1}{3} + \frac{1}{18} + \frac{1}{108} + \frac{1}{648} + \cdots$

$$\times\frac{1}{6} \qquad \times\frac{1}{6} \qquad \times\frac{1}{6} \qquad \times\frac{1}{6}$$

Geometric : $a = 2$, $r = \frac{1}{6}$

since $\left|\frac{1}{6}\right| < 1$, series converges

$$\text{sum} = \frac{a}{1-r} = \frac{2}{1-\frac{1}{6}} = \frac{2}{(5/6)} = \frac{12}{5}$$

Ex let's do $\displaystyle\sum_{i=1}^{\infty} \frac{10^i}{(-9)^{i-1}}$

It is geometric : first term: plug in $i=1$, get $a = \frac{10}{1} = 10$

common ratio : put $a_i = \frac{10^i}{(-9)^{i-1}}$

and look at $\frac{a_{i+1}}{a_i} = \frac{10^{i+1}}{(-9)^i} \Big/ \frac{10^i}{(-9)^{i-1}} = \frac{10}{-9} = -\frac{10}{9}$

ratio $= \frac{-10}{9}$, and $\left|-\frac{10}{9}\right| > 1$, so series diverges.

Ex Geometric series explains a bit of old math

$$1.\overline{73} = 1.737373\cdots$$

is actually a representation of a series

$$1.737373\cdots$$

$$= 1 + 0.73 + 0.0073 + 0.000073$$

$$= 1 + \underbrace{\frac{73}{100} + \frac{73}{100^2} + \frac{73}{100^3} + \cdots}_{\text{geometric series with}} \qquad a = \frac{73}{100}, \quad r = \frac{1}{100}$$

$$= 1 + \sum_{i=1}^{\infty} \frac{73}{100}\left(\frac{1}{100}\right)^{i-1}$$

$$= 1 + \frac{a}{1-r} = 1 + \frac{73/100}{1 - 1/100} = 1 + \frac{73/100}{99/100} = 1 + \frac{73}{99} = \frac{172}{99}$$

$\rightarrow$ repeating decimal can be written as a fraction.

<u>Ex</u> Consider the series $\displaystyle\sum_{i=0}^{\infty} \frac{(x+3)^i}{2^i}$

For what values of x does it converge?

The series is geometric with $r = \dfrac{x+3}{2}$

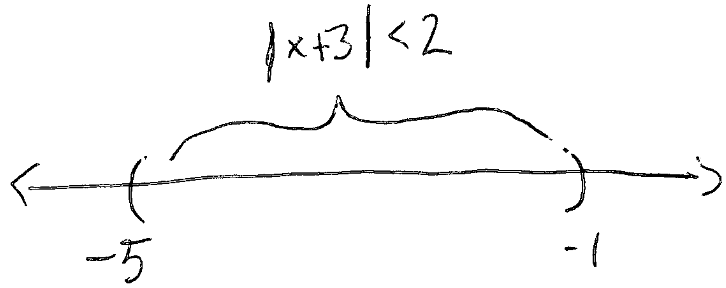
It converges when $|r| < 1$

$$\left|\frac{x+3}{2}\right| < 1$$

$$\text{or } |x+3| < 2$$

To find the <u>endpoints</u> of this region, we solve $|x+3| = 2 \Rightarrow x+3 = 2$ or $x+3 = -2$ $\Rightarrow x = -1$ or $x = -5$

So the region is $-5 < x < -1$ or $x \in (-5,-1)$



endpoints not included

Test for divergence

If $\sum_{i=1}^{\infty} a_i$ converges, then $\lim_{n\to\infty} a_n = 0$.

Contrapositive: If $\lim_{n\to\infty} a_n$ doesn't exist

or $\lim_{n\to\infty} a_n$ exists but isn't 0

then $\sum_{i=1}^{\infty} a_i$ diverges

Ex $\sum_{i=1}^{\infty} \frac{3i+4}{4i-7}$ diverges $\left(\text{since } \lim_{i\to\infty} \frac{3i+4}{4i-7} = \frac{3}{4} \neq 0\right)$

Intuition: can't keep adding big terms and expect series to converge, terms have to get small.

Even if the terms do get smaller and smaller convergence of the series is not assured!

<u>Ex</u> (The Harmonic Series)

$$a_i = \frac{1}{i} \qquad \sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

check $\lim_{i \to \infty} a_i = \lim_{i \to \infty} \frac{1}{i} = 0$

since a_i converges to 0, $\sum a_i$ <u>might</u> converge.

BUT NO! $\sum_{i=1}^{\infty} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ DIVERGES

In fact if $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is nth partial sum,

then $S_{2^n} > 1 + \frac{n}{2} \to \infty$.

$\boxed{\text{Easier proof next time ...}}$