Sequences

A *sequence* is a list of numbers in a definite order:

$$a_1 \quad a_2 \quad a_3 \quad \cdots \quad a_{100} \cdots a_{2875} \cdots$$

$$1 \quad 2 \quad 3 \quad \cdots \quad 100 \cdots 2875 \cdots$$

Notations $\{a_1, a_2, \dots\}$

$\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$ (To make range of n explicit)

Ex $a_n = n^2$ (Explicit function of n)

$a_1 = 1$, $a_2 = 4$, $a_3 = 9, \dots, a_{32} = 1024, \dots$

Ex Fibonacci sequence (recursively defined)

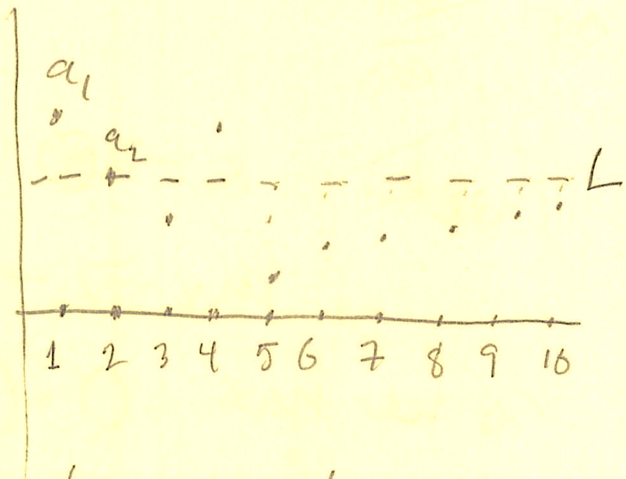
$f_1 = 1$, $f_2 = 1$, $f_3 = f_1 + f_2 = 2$, $f_4 = f_2 + f_3 = 3$

generally; $f_n = f_{n-1} + f_{n-2}$

Ex An empirical sequence

$a_n =$ number of people who attend the nth calculus lecture

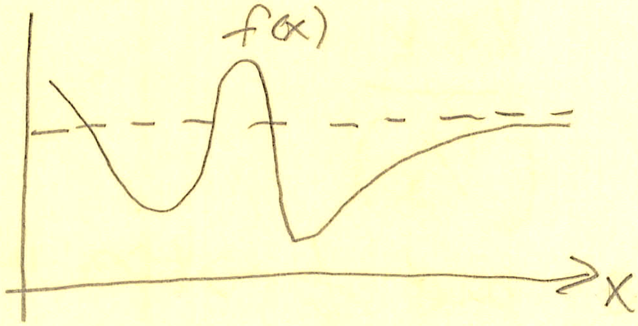
Limits of sequences



$$\lim_{n\to\infty} a_n = L$$

(CONVERGES)

Analogous to



$$\lim_{x\to\infty} f(x) = L$$

(Horizontal Asymptote)

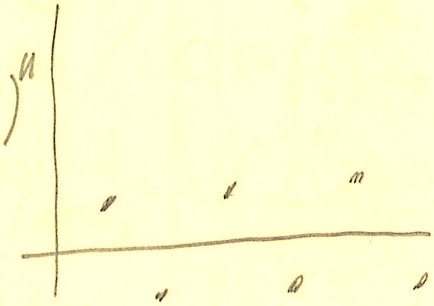
other possibilities

$a_n = n$

$$\lim_{n\to\infty} a_n = \infty$$

(DIVERGES TO ∞)

$a_n = (-1)^n$



$\lim_{n\to\infty} a_n$ does not exist

(DIVERGES)

Sequences are analogous to functions

(In fact, "a sequence is a function whose domain is the set of positive integers" $\{1, 2, \ldots\}$)

If we can write $a_n = f(n)$, where f is a function of a real variable,

and $\lim_{x\to\infty} f(x)$ exists,

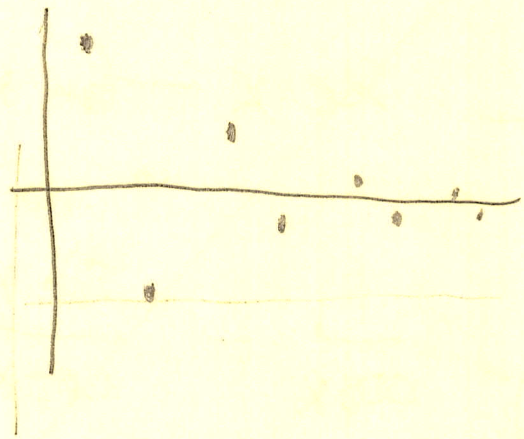
Then $\lim_{n\to\infty} a_n$ exists

and $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$.

Ex $a_n = e^{-n}$ $\quad \lim_{n\to\infty} e^{-n} = \lim_{x\to\infty} e^{-x} = 0$.

* We maintain the distinction between sequences and functions

Ex $a_n = \frac{(-1)^n}{n}$



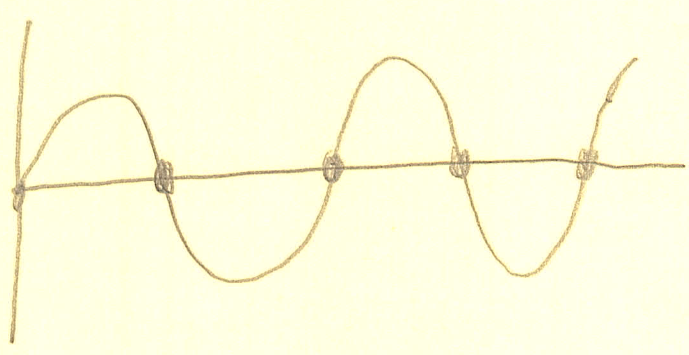
$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

But $(-1)^x$ does not make sense as a function of a real variable x.

Ex $a_n = \sin(\pi n)$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin(\pi n) = 0$$

Because $\sin(\pi n) = 0$ for every integer n.



[Even though $\lim_{x\to\infty} \sin(\pi x)$ oscillates/does not exist]

Ex $a_n = \frac{\ln n}{n}$ ⇝ switch to function $f(x) = \frac{\ln x}{x}$

$$\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{(1/x)}{1} \quad \text{(L'Hospital)}$$

$$= \lim_{x \to \infty} \frac{1}{x} = 0.$$

So $\lim_{n \to \infty} \frac{\ln n}{n} = 0$ as well.

Useful facts:

Squeeze theorem:

If $a_n \leq b_n \leq c_n$

and $\lim_{n \to \infty} a_n = L = \lim_{n \to \infty} c_n$

then $\lim_{n \to \infty} b_n = L$ as well.

Corollary: If $\lim_{n \to \infty} |a_n| = 0$, then $\lim_{n \to \infty} a_n = 0$

(since $-|a_n| \leq a_n \leq |a_n|$).

Ex

$$\lim_{n\to\infty} \frac{n!}{n^n} = ?$$

$$\frac{n!}{n^n} = \frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdot \frac{4}{n} \cdot \dots \cdot \frac{n}{n} = \frac{1}{n}(\text{something} < 1)$$

So $\frac{n!}{n}$ is squeezed between 0 and $\frac{1}{n}$

since $\lim\limits_{n\to\infty} \frac{1}{n} = 0$, $\lim\limits_{n\to\infty} \frac{n!}{n^n} = 0$ as well.

Ex $a_n = \dfrac{(\cos n)(-1)^n}{\sqrt{n}}$

Since $-1 \le \cos n \le 1$, we have $|\cos n| \le 1$

Thus $|a_n| \le \dfrac{|\cos n|\ |(-1)^n|}{|\sqrt{n}|} = \dfrac{1}{\sqrt{n}}$

Since $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, $\lim\limits_{n\to\infty} |a_n| = 0$ as well

and furthermore $\lim\limits_{n\to\infty} a_n = 0$ by the corollary.

If $\lim_{n\to\infty} a_n = L$ converges and $f(x)$ is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$

Ex $\lim_{n\to\infty} \cos\left(\frac{(-1)^n}{n}\right) = \cos\left(\lim_{n\to\infty} \frac{(-1)^n}{n}\right) = \cos(0) = 1$

Definitions $\{a_n\}$ is increasing if $a_{n+1} \geq a_n$

$\{a_n\}$ is decreasing if $a_{n+1} \leq a_n$

$\{a_n\}$ is monotonic if either increasing or decreasing

$\{a_n\}$ is bounded if $A \leq a_n \leq B$ for some constants A and B.

Theorem A bounded, monotonic sequence CONVERGES!

Problem Prove $\{a_n\}$ converges and find the limit

$$a_1 = 2 \qquad a_{n+1} = \frac{1}{2}(a_n + 1)$$

Lemma $1 < a_n$ for all n.

Proof induction on n. Base case $a_1 = 2 > 1$

Suppose $1 < a_n$ for some $n > 1$

then $2 < a_n + 1$, and $1 < \frac{1}{2}(a_n + 1) = a_{n+1}$

so $1 < a_{n+1}$ as well.

Corollary: $a_{n+1} < a_n$

Proof since $1 < a_n$, we have $a_n + 1 < 2a_n$

and $\frac{1}{2}(a_n + 1) < a_n$

Lemma says $\{a_n\}$ is bounded below.

Corollary says $\{a_n\}$ is decreasing.

Monotonic Convergence theorem applies,

and $\{a_n\}$ converges. $\lim_{n\to\infty} a_n = L$

we know $L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n + 1) = \frac{1}{2}(L+1)$

$\Rightarrow L = 1$