Review for final exam

Office Hours Monday 10-12

Oscar's Review session 12/6 4-6pm
RLM 7.104

L'Hospital's rule $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$ or $\frac{0}{0}$

and if $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists

then $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Improper integrals

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

Main place that improper integrals come up is the integral test for convergence of a series

Sequences $\{a_n\}$ $\lim_{n\to\infty} a_n$

Main theorem: bounded monotonic sequence converges.

Series $\sum_{i\geq 1}^{\infty} a_i$ converges if the sequence of partial sums converges

Tests 1) $\sum \frac{1}{n^p}$ converges iff $p>1$
diverges if $p\leq 1$

2) $\sum_{n=1}^{\infty} ar^{n-1}$ converges if $|r|<1$
diverge if $|r|\geq 1$

$$= \frac{a}{1-r}$$

3) Comparison positive terms $a_n \leq b_n$

$\sum a_n$ diverges $\Rightarrow$ $\sum b_n$ diverges

$\sum b_n$ converge $\Rightarrow$ $\sum a_n$ converges

3') limit comparison positive a_n, b_n

If $\lim \frac{a_n}{b_n} = C$ is finite >0 both converge or diverge

4) Check if $\lim a_n = 0$ — Test for divergence

otherwise $\sum a_n$ diverges

5) $\sum (-1)^{n-1} b_n \quad b_n \geq 0$

Need $b_{n+1} \leq b_n \qquad \lim_{n \to 0} b_n = 0$

$\Rightarrow$ converges by Alternating series test

6) $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$

$L < 1 \Rightarrow \sum a_n$ converges (absolutely)

$L > 1 \Rightarrow \sum a_n$ diverges

$L = 1$ inconclusive

7) $\lim_{n \to \infty} \sqrt[n]{|a_n|} = L \qquad L < 1$ absolute convergence

$L > 1$ divergence

$L = 1$ inconclusive

8) Integral test $a_n = f(n)$

$f(x)$ is positive continuous and decreasing

$\sum a_n$ converges $\Leftrightarrow$ $\int_1^\infty f(x)\,dx$ converges.

Power Series, Taylor Series

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

center

it has an interval of convergence $(a-R, a+R)$

R = radius of convergence

Use ratio test to find R

Need to use other test to check the endpoints $a-R$, $a+R$

might get $[a-R, a+R)$

Taylor series $f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

eg. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Taylor's theorem: consider interval of x such that $|x-a| \leq d$

suppose we find M s.t. $M \geq |f^{(n+1)}(x)|$ for $|x-a| \leq d$

THEN

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$$

$f(x) - T_n(x)$

n-th degree Taylor polynomial

Then we did Parametric equations

$x = f(t)$
$y = g(t)$

$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j}$

2D vector

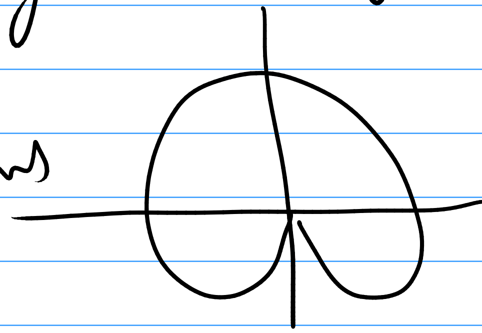
slopes $\frac{dy}{dx} = (dy/dt)/(dx/dt)$

Tangent lines and arc length

Polar coords $(x,y) = (r\cos\theta, r\sin\theta)$

$r^2 = x^2 + y^2$ $\qquad \tan\theta = \frac{y}{x}$

Polar equations



$r = 1 + \sin\theta$

$\Big\{$ Area and arclength of polar curves

$\uparrow$ double integrals

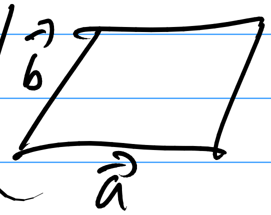
$\nwarrow$ arclength of parametric curve

$x = r(\theta)\cos\theta$
$y = r(\theta)\sin\theta$

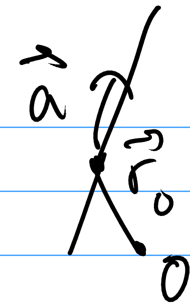
2D & 3D vectors addition, scalar multiplication

dot product $\leftarrow$ angles, projections

cross product $\leftarrow$ areas of parallelogram
right hand rule gives the direction

A() $= |\vec{a} \times \vec{b}|$

lines and planes



Line $\vec{r}(t) = \vec{r}_0 + t\vec{a}$

Plane $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$

normal vector

general point on the plane

a fixed point on the plane

$\vec{n} = \langle a, b, c \rangle$ $\quad\quad$ $\vec{r} = \langle x, y, z \rangle$

$\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

→ Aside on quadric surfaces

Vector-valued functions

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

$$\vec{v} = \vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

$\vec{v}$ = velocity or tangent vector

can get tangent lines.

Arc length $\int_a^b |\vec{r}\,'(t)|\,dt$

Functions of several variables $f(x,y)$ $f(x,y,z)$

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rightarrow \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

gradient

Tangent plane to the graph $z = f(x,y)$

$$z - z_0 = \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0)$$

at (x_0, y_0)

Linear approximation

$z = f(x,y)$ $z_0 = f(x_0, y_0)$

$$f(x,y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0)$$

$F(x,y,z) = f(x,y) - z$ $\nabla F = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1 \right\rangle$

$$\nabla F(x_0, y_0, z_0) \cdot (\vec{r} - \vec{r}_0) = 0$$

So tangent plane to graph is also tangent plane to a level set of F

$\nabla f = \langle f_x, f_y, f_z \rangle$ or $\langle f_x, f_y \rangle$

Direction of gradient = steepest increase

Directional derivative

$$D_{\vec{u}} f = \lim_{h \to 0} \frac{f(\vec{r} + h\vec{u})}{h} = \nabla f \cdot \vec{u}$$

$$= |\nabla f| |\vec{u}| \cos\theta$$

Chain rule

$$\begin{tikzcd}[column sep=small] & f \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ x \arrow[d, dash] & y \arrow[d, dash] & z \arrow[d, dash] \\ t & t & t \end{tikzcd}$$

$f(x, y, z)$

x, y, z functions of t

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$D_{\vec{u}} f = \nabla f \cdot \vec{u}$ means that ∇f is perpendicular to the level sets $f(x,y,z) = k$ of f

Max/Min (2D) Critical point $\nabla f = 0$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

Max	$D > 0$	$f_{xx} < 0$
Min	$D > 0$	$f_{xx} > 0$
Saddle	$D < 0$	

GLOBAL Max/Min need to check the boundary of the domain

Lagrange Multipliers

Optimize $f(x,y,z)$ subject to $g(x,y,z)=k$

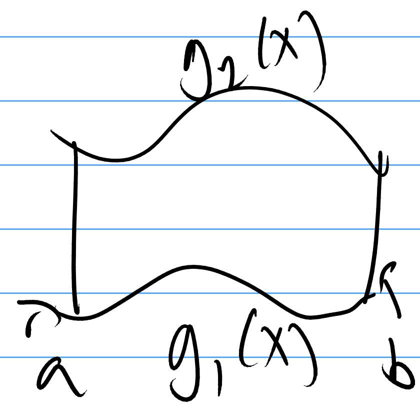
$$\left\{ \begin{array}{l} \nabla f = \lambda \nabla g \\ g(x,y,z) = k \end{array} \right\}$$

4 equations in 4 unknowns

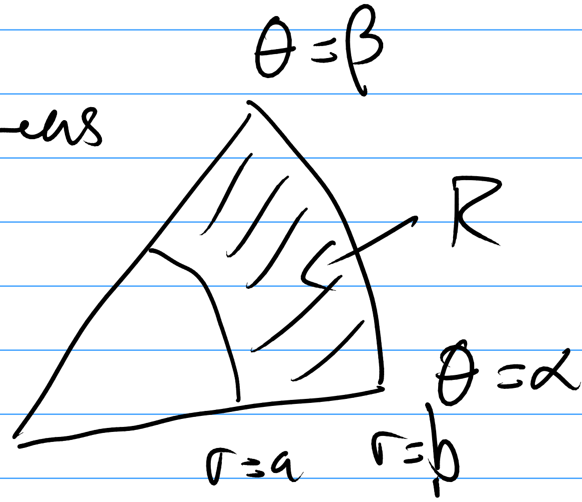
Double (Triple) Integrals

$$\iint_{R=[a,b]\times[c,d]} f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

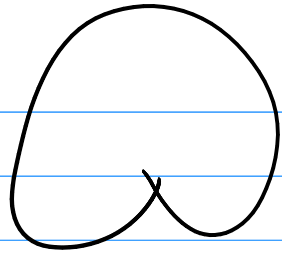


Polar areas



$$\iint_R f\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

can replace a and b with $r_1(\theta)$, $r_2(\theta)$

$A\Big($  $\Big) = \int_0^{2\pi}\int_0^{1+\sin\theta} 1 \, r\,dr\,d\theta$

$r = 1+\sin\theta$

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

*** For double and triple int s, really important to get limits of integration right.

Change of variables

$x = g(u,v) \qquad y = h(u,v)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\iint_R f(x,y)\,dA = \iint_S f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Provided R is the image of S.