# Absolute convergence, root & ratio tests

Mentioned Absolute convergence last time

Definition A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Ex $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent since

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$ is convergent.

Ex $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is not absolutely convergent because

$$\sum_{n=1}^{\infty} \frac{1}{n}$$ is divergent.

However, alternating series test implies $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent.

Def A series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if it is convergent but not absolutely convergent.

Also mentioned last time:

Theorem (Absolute convergence implies convergence)

If a series $\sum a_n$ is absolutely convergent then it is convergent.

Proof: Start with $-|a_n| \leq a_n \leq |a_n|$

add $|a_n|$ $\qquad 0 \leq a_n + |a_n| \leq 2|a_n|$

* assumption that $\sum a_n$ is absolutely convergent means that $\sum |a_n|$ is convergent.
* Hence $\sum 2|a_n|$ is convergent.
* Since $\sum a_n + |a_n|$ is a series of positive terms less than $\sum 2|a_n|$, which converges, the comparison test implies that $\sum (a_n + |a_n|)$ converges.

* Now $\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$ is the difference of two convergent series, and hence is convergent.

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n^3} = \frac{\cos(1)}{1} + \frac{\cos(2)}{2^3} + \frac{\cos(3)}{3^3} + \cdots$

series has both positive and negative terms, but is not alternating

$\cos 1 > 0$, $\cos 2 < 0$, $\cos 3 < 0$, etc.

Look for absolute convergence:

$\displaystyle\sum_{n=1}^{\infty} \frac{|\cos(n)|}{n^3}$ can compare with $\frac{1}{n^3} \geq \frac{|\cos n|}{n^3}$

since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, we see that $\displaystyle\sum_{n=1}^{\infty} \frac{|\cos n|}{n^3}$ converges by comparison test. So $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^3}$ converges as well.

$$\underset{\text{converges}}{\sum_{n=1}^{\infty} \frac{1}{n^3}} \xRightarrow{\text{comparison}} \underset{\text{converges}}{\sum_{n=1}^{\infty} \frac{|\cos n|}{n^3}} \xRightarrow[\text{conv.}]{\text{absolute}} \underset{\text{converges}}{\sum_{n=1}^{\infty} \frac{\cos n}{n^3}}$$

The next two tests are actually tests for absolute convergence.

<u>Theorem</u> (Ratio Test) Consider $\sum_{n=1}^{\infty} a_n$

(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and hence convergent).

(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$ then $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, no conclusion can be drawn, and other tests must be used.

<u>Ex</u> $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$. $a_n = \frac{n^2}{2^n}$: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^2}{2^{n+1}} \Big/ \frac{n^2}{2^n}\right|$

$= \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \frac{2^n}{2^{n+1}} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \frac{1}{2} = \lim_{n\to\infty} \frac{1}{2}\left(1+\frac{1}{n}\right)^2$

$= \frac{1}{2}$ so $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ <u>converges absolutely</u>.

Theorem (Root Test) Consider $\sum_{n=1}^{\infty} a_n$

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and hence convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or the limit is $\infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, no conclusion can be drawn, and other tests must be used

Ex $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ $\quad \sqrt[n]{|a_n|} = \sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n} = \frac{2n+3}{3n+2}$

$$\lim_{n\to\infty} \frac{2n+3}{3n+2} = \lim_{n\to\infty} \frac{2+\frac{3}{n}}{3+\frac{2}{n}} = \frac{2}{3} < 1$$

so the series converges by the root test.

The ratio and root tests are related on a theoretical level: both are designed to detect the exponential growth or decay of the terms of the series. That is, they detect the extent to which $\sum_{n=1}^{\infty} a_n$ resembles a geometric series $\sum_{n=1}^{\infty} ar^{n-1}$.

For example, if we apply the ratio and root tests to the geometric series, we find

$$\lim_{n\to\infty} \frac{|ar^n|}{|ar^{n-1}|} = |r|$$

$$\lim_{n\to\infty} \sqrt[n]{|ar^{n-1}|} = \lim_{n\to\infty} \sqrt[n]{\frac{|a|}{|r|}}\ |r| = |r|$$

$$\left(\text{since } \lim_{n\to\infty} \sqrt[n]{|c|} = 1\right)$$

This reveals the meaning of the limit in the ratio and root tests.

<u>NOTE</u>: If ratio test is inconclusive, root test will also be inconclusive, and vice versa.

Ex $\displaystyle\sum_{n=0}^{\infty} \frac{(-10)^n}{n!} = \frac{(-10)^0}{0!} + \frac{(-10)^1}{1!} + \frac{(-10)^2}{2!} + \frac{(-10)^3}{3!}$

$= 1 + (-10) + 50 + \left(-\frac{1000}{6}\right)$

Ratio test tends to simplify factorials:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{10^{n+1}}{(n+1)!} \Big/ \frac{10^n}{n!} = \lim_{n\to\infty} \frac{10^{n+1}}{10^n} \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} 10 \cdot \frac{1}{n+1} = 0 < 1$$

So the series converges absolutely, and hence converges.

Ex $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$

whole term is a power, so try root test.

$$a_n = \left(1+\frac{1}{n}\right)^{n^2}, \quad \sqrt[n]{|a_n|} = \left[\left(1+\frac{1}{n}\right)^{n^2}\right]^{1/n}$$

$$\text{so } \sqrt[n]{|a_n|} = \left(1+\frac{1}{n}\right)^{n^2/n} = \left(1+\frac{1}{n}\right)^{n}$$

Now we have

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e > 1$$

So the series diverges by the root test.