Cylinders & Quadrics

But First: Exam 2 Review

* Taylor Series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

* Taylor's Theorem:

If $\left|f^{(n+1)}(x)\right| \leq M$ for $|x-a| \leq d$

then $\left|R_n(x)\right| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$ for $|x-a| \leq d$

where $R_n(x) = f(x) - T_n(x)$ is the remainder of the degree n Taylor polynomial.

* operations with power series
* Differentiation and integration of power series.
* Approximation with Taylor polynomials

* some basic Taylor series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{if } |x|<1 \quad \text{etc.}$$

* Parametric curves, tangents, arc length, area, circular motion
* Same things for polar curves.
* Vectors
  - dot products, angles, projections
  - cross products, relation to areas and volumes.

New material starts now:

Last time we discussed planes, which are defined by linear equations

Today we will be focused on surfaces defined by quadratic equations, called quadric surfaces.

Here we are dabbling in a subject called ALGEBRAIC GEOMETRY, the study and classification of shapes according to the algebraic properties of their defining equations.

The most general quadratic equation in 3 variables has the form

$$0 = Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J$$

where $A, B, C, D, E, F, G, H, I, J$ are some coefficients.

We can simplify this equation by changing coordinates:

Eg: $xy + z^2 = 1$

set $X = x - y$ $\quad Y = x + y$

then $X^2 - Y^2 = x^2 - 2xy + y^2 - x^2 - 2xy - y^2$

$= -4xy$

so $xy = \frac{1}{4}Y^2 - \frac{1}{4}X^2$

$xy + z^2 = 1 \iff -\frac{1}{4}X^2 + \frac{1}{4}Y^2 + z^2 = 1$

Got rid of the cross term!

FACT: using manipulations such as this, any quadratic equation can be brought into one of the forms

(a) $Ax^2 + By^2 + Cz^2 + J = 0$

(b) $Ax^2 + By^2 + Iz = 0$

Now we will classify the possible surfaces that can arise in each of these cases.

Case (a):

Suppose A, B, C are all positive, and J is negative. Then we can write the equation as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\left(\text{e.g. } \frac{1}{a^2} = \frac{A}{-J} \right)$$

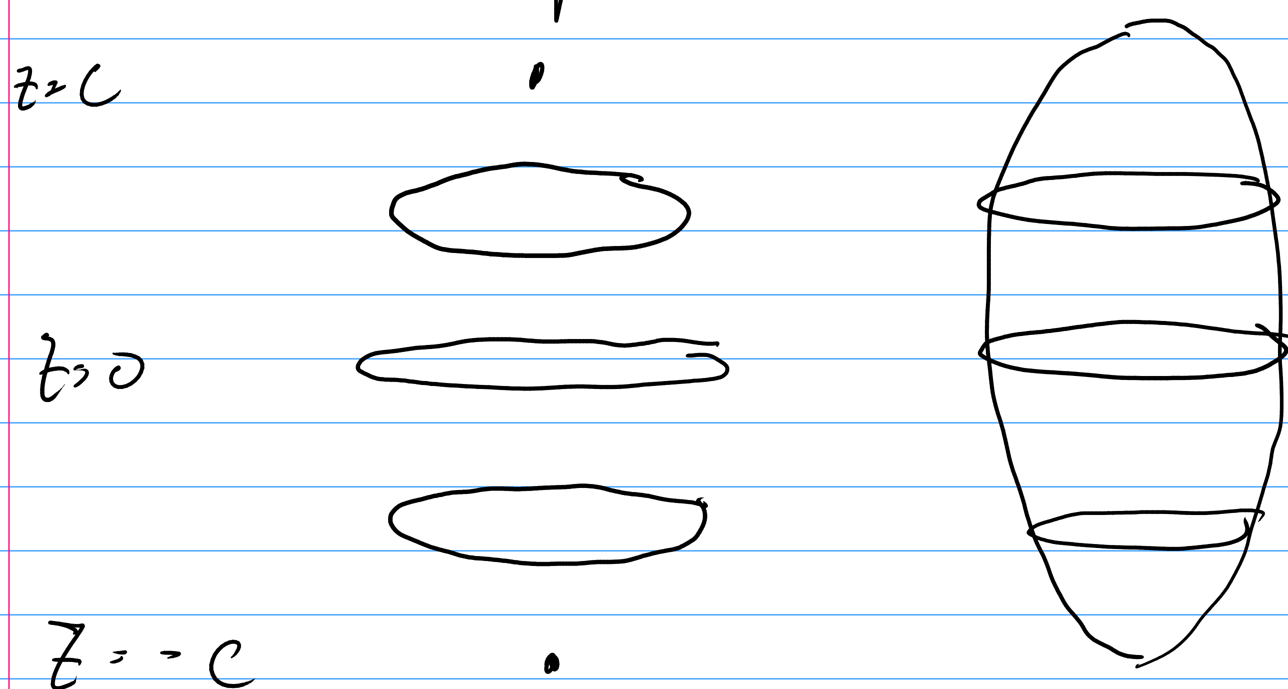
What shape is it?

Idea: use "traces", that is, intersections with planes of the form $x = \text{constant}$
$y = \text{constant}$
$z = \text{constant}$.

If we hold $z = z_0$ constant in the equation, we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(1 - \frac{z_0^2}{c^2}\right)$$

which is the equation of a 2d ellipse, whose size depends on z_0:



Overall shape is an ellipsoid;
(like a sphere but oblong).

Next case $A>0$, $B>0$, $C<0$

Can write as either

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

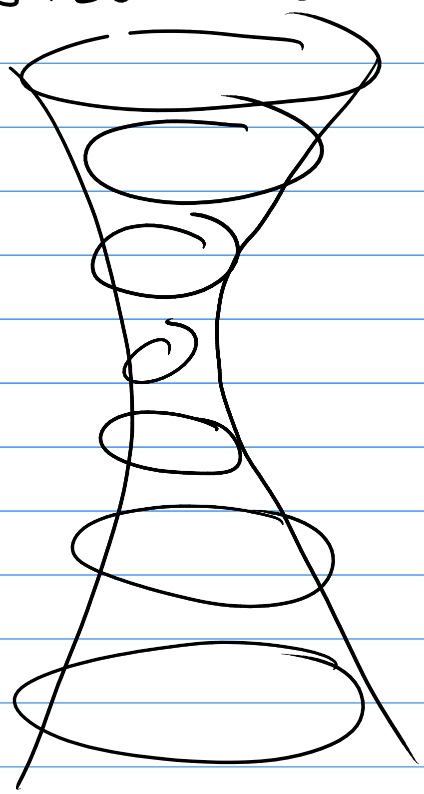
OR $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0 \iff \frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$

OR $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1 \iff \frac{-x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

First case

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

$z=$ const trace is an ellipse, whose size increases as $z \to \pm\infty$



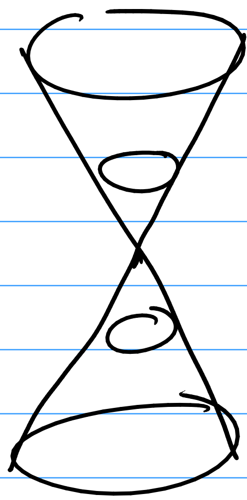
Hyperboloid of one sheet

The $x=$ const traces are hyperbolas.

Second case

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$z=$ const trace is a ellipse, unless $z=0$, in which case it is a point.



Cone

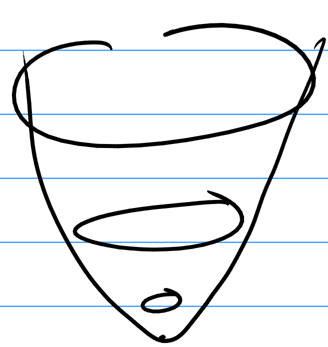
The $x=$ const traces are hyperbolas, except $x=0$, which is two lines.

Third case

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

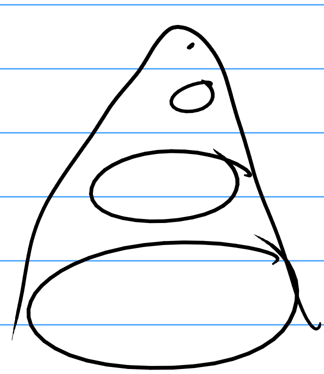
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}-1$$

$z=$ const traces are ellipses, except when $|z|<c$, in which case they are empty



Hyperboloid of two sheets

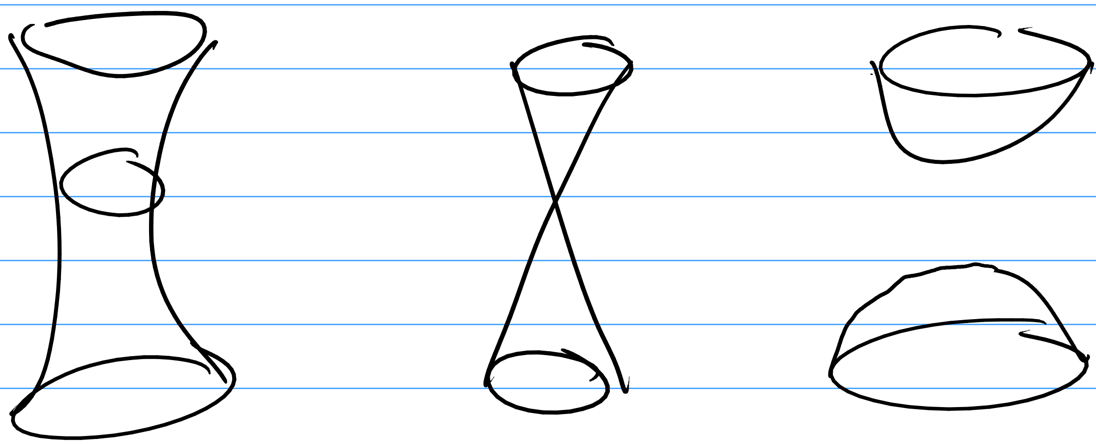
$x=$ const traces are hyperbolas.



Another perspective on these three cases

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = d$$

$d > 0$ $\qquad$ $d = 0$ $\qquad$ $d < 0$



"Topological Transition"

"pinching a neck"

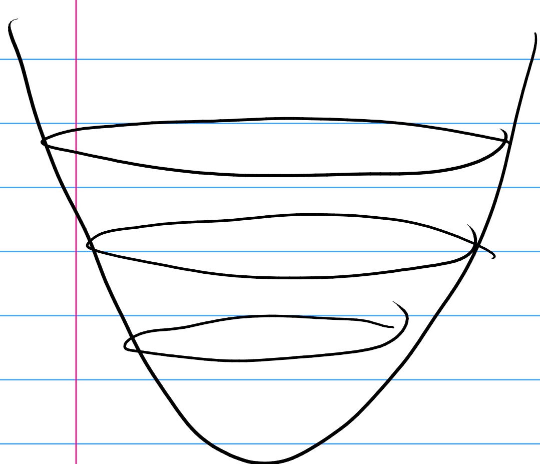
Case (b) $Ax^2 + By^2 + Iz = 0$

If $A > 0$, $B > 0$, can write

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$

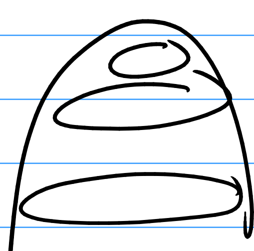
Assume $c > 0$. The $z = $ const traces are ellipses if $z > 0$, empty if $z < 0$.

The $x = $ const traces are parabolas



Elliptic Paraboloid.

(If $c < 0$



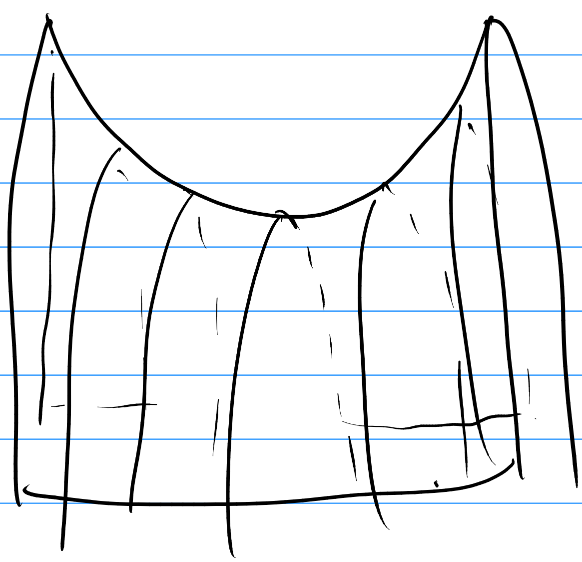
also elliptic paraboloid)

If $A > 0$ and $B < 0$

$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

$z =$ const traces are hyperbolas

$x =$ const traces are parabolas



Hyperbolic paraboloid

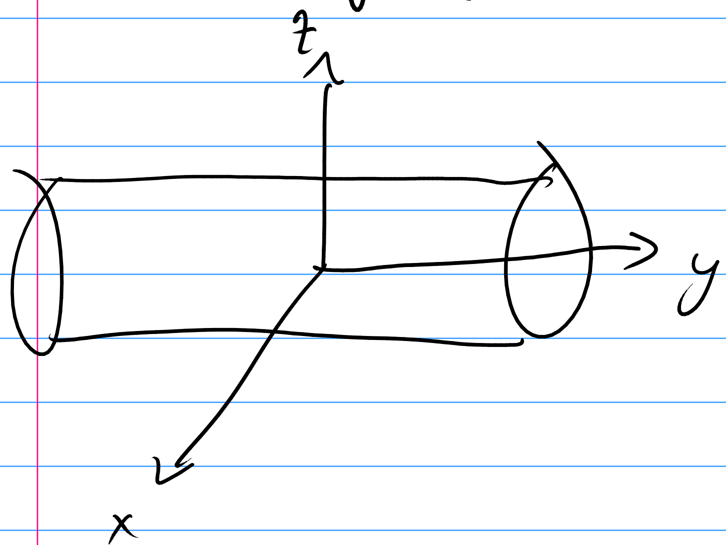
also called

a "saddle"

If some square terms are missing:

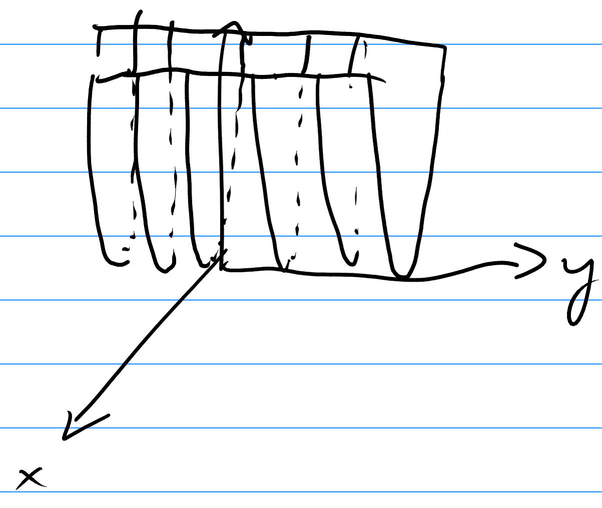
e.g. $x^2 + \cancel{0 \cdot y^2} + z^2 = 1$

just get the circle $x^2 + z^2 = 1$, with any y coordinate.



a cylinder

Or $z = x^2 + 0y^2$



a parabolic cylinder

If $f(x,y)=0$ is a curve in the plane, then

$\{ f(x,y)=0, \ z = \text{anything} \}$ in space

is a "cylinder over $f(x,y)=0$".

$\{ x^2 - y^2 = 1, \ z = \text{anything} \}$

is a "hyperbolic cylinder", etc.

Given an equation, how do you classify it?

Useful trick is completing the square

$$x^2 + bx = \left(x + \tfrac{b}{2}\right)^2 - \left(\tfrac{b}{2}\right)^2$$

Ex Classify the surface

$$x^2 - y^2 + z^2 - 4x - 2y - 2z + 4 = 0$$

$$x^2 - 4x = (x-2)^2 - 2^2$$

$$-y^2 - 2y = -[y^2 + 2y] = -[(y+1)^2 - 1^2]$$

$$z^2 - 2z = (z-1)^2 - 1^2$$

Get $(x-2)^2 - 4 - [(y+1)^2 - 1] + (z-1)^2 - 1 + 4 = 0$

$$(x-2)^2 - (y+1)^2 + (z-1)^2 = 0$$

Define $X = x-2 \quad Y = y+1 \quad Z = z-1$

$$X^2 - Y^2 + Z^2 = 0$$

This is a cone that opens along the y-axis.