Power Series

Series of functions, that is, series that contain the variable x.

Consider the series of the form.

$$\sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$$

C_n = coefficients

x = variable

Function of x, and it can converges for some values of x, but not others.

$$\sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1 (x-a) + C_2 (x-a)^2 + \cdots$$

power series centered at a.

$(x-a)^n$

For what x does $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge?

ratio test:

$a_n = \frac{x^n}{n!}$ $\quad \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \Big/ \frac{x^n}{n!}\right|$

$= \lim_{n\to\infty} x \frac{n!}{(n+1)!} = \lim_{n\to\infty} x \frac{1}{n+1}$ $\quad$ b/c $(n+1)! = (n+1)n!$

$= 0$ no matter what x is.

So $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all $\underbrace{x \text{ in } (-\infty, \infty)}_{-\infty < x < \infty}$

(preview $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$)

For what x does $\sum_{n=0}^{\infty} n!\, x^n$ converge?

for $x=0$: $\sum_{n=0}^{\infty} n!\, 0^n = \underbrace{0!\, 0^0}_{1} + \underbrace{1!\, 0^1}_{0} + \cdots$

$0! = 1$ and for the purposes of power series, $0^0 = 1$

what if $x \neq 0$?

$$\lim_{n\to\infty} \frac{(n+1)!\, x^{n+1}}{n!\, x^n} = (n+1)x = \infty$$

diverges for $x \neq 0$.

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n} \qquad \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{n+1} \Big/ \frac{(x-3)^n}{n} \right|$$

$$= \lim_{n\to\infty} \left| (x-3)\frac{n}{n+1} \right| = |x-3| \underbrace{\lim_{n\to\infty} \frac{n}{n+1}}_{=1}$$

$$= |x-3|$$

if $|x-3|<1$ the series converges

if $2<x<4 \Rightarrow$ convergence $(2,4)$

if $|x-3|>1$ the series diverges

if $x<2$ or $x>4 \Rightarrow$ divergence.

$(-\infty,2) \cup (4,\infty)$

what about $|x-3|=1$ that is $x=2$ or $x=4$.

$x=2$ get $\sum_{n=1}^{\infty} \frac{(2-3)^n}{n} = \sum \frac{(-1)^n}{n}$ converges

$x=\cancel{3}4$ get $\sum_{n=1}^{\infty} \frac{(4-3)^n}{n} = \sum \frac{1}{n}$ diverges

Final answer $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converges

if and only if $2 \le x < 4$

x in $\underbrace{[2,4)}$

this is called the <u>interval of convergence</u>

<u>Theorem</u> let $\sum_{n=0}^{\infty} c_n (x-a)^n$ be a power series centered at a. There are 3 possibilities

(i) the series converges only at $x=a$.

(ii) the series converges for all x.

(iii) there is a positive number R such that

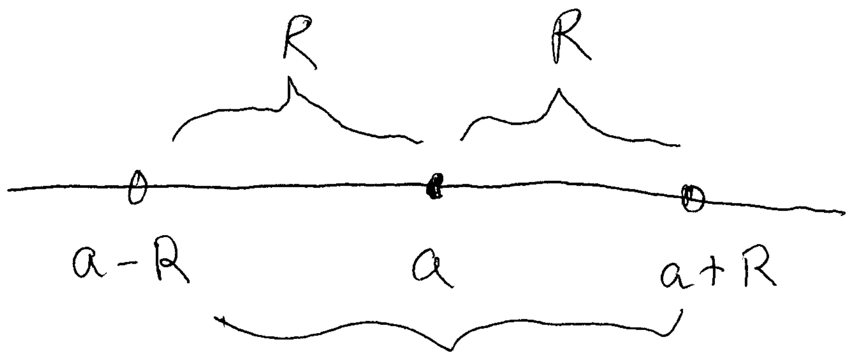
if $|x-a| < R$, the series converges

if $|x-a| > R$, the series diverges

R is called the <u>radius of convergence</u>

In case (iii) Need to check endpoints $|x-a| = R$

	Radius of convergence	interval of conv.
$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$R = \infty$	$(-\infty, \infty)$
$\sum_{n=0}^{\infty} n!\,x^n$	$R = 0$	$\{0\}$
$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$



$|x-a| < R$

Ex $\sum_{n=1}^{\infty} \frac{x^n}{5^n n^5}$

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{5^{n+1}(n+1)^5} \Big/ \frac{x^n}{5^n n^5} \right| = \lim_{n\to\infty} \frac{|x|}{5} \frac{n^5}{(n+1)^5}$$

$$= \frac{|x|}{5} \underbrace{\lim_{n\to\infty} \frac{n^5}{(n+1)^5}}_{1} = \frac{|x|}{5} < 1 \Rightarrow \text{convergence}$$

means $|x| < 5$

Radius of convergence is 5, centered at 0.

endpoints : $x = 5$ or $x = -5$

$x = 5$: $\sum_{n=1}^{\infty} \frac{5^n}{5^n n^5} = \sum_{n=1}^{\infty} \frac{1}{n^5}$ converges by p-test.

$x = -5$ $\sum_{n=1}^{\infty} \frac{(-5)^n}{5^n n^5} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$ converges.

interval of convergence = $[-5, 5]$

Ex $\sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n$

$$\left| \frac{(-(n+1))^n}{(n+1)!} x^{n+1} \Big/ \frac{(-n)^{n-1}}{n!} x^n \right|$$

$$= \frac{(n+1)^n}{n^{n-1}} \frac{n!}{(n+1)!} |x| = \frac{(n+1)^n}{n^{n-1}} \cdot \frac{1}{n+1} |x|$$

$$= \frac{(n+1)^{n-1}}{n^{n-1}} |x| = \left(\frac{n+1}{n}\right)^{n-1} |x| = \left(1 + \frac{1}{n}\right)^{n-1} |x|$$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n-1} |x| = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n}\right)^{-1} |x|$$

$$= e \cdot 1 \cdot |x| < 1 \iff |x| < \frac{1}{e}$$

Radius of convergence is $\frac{1}{e}$.

what about endpoints $x = -\frac{1}{e}$

$\sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} \left(-\frac{1}{e}\right)^n$?

First person to solve this wins \$1.

Ex $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$ can use root test

~~$|x|$~~ $\sqrt[n]{\left|\frac{x^n}{n^n}\right|} = \frac{|x|}{n}$

$\lim_{n\to\infty} \frac{|x|}{n} = 0$ nomatter what x is.

Radius of convergence $= \infty$

converges for all x in $(-\infty, \infty)$