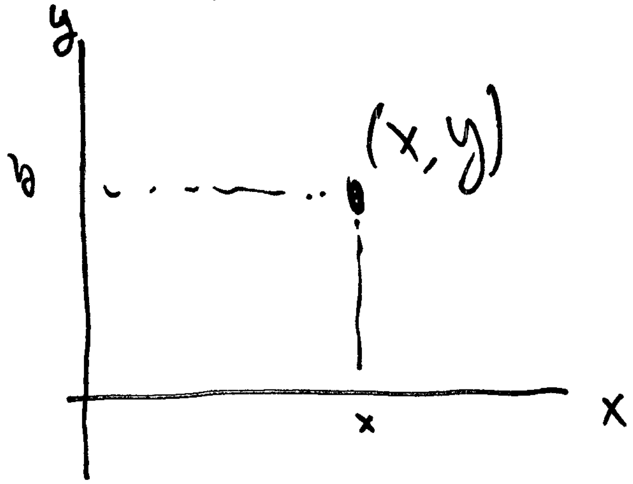
# Polar coordinates

What are cartesian coordinates?




x = distance from the point to the y-axis

y = distance from the point to the x-axis.

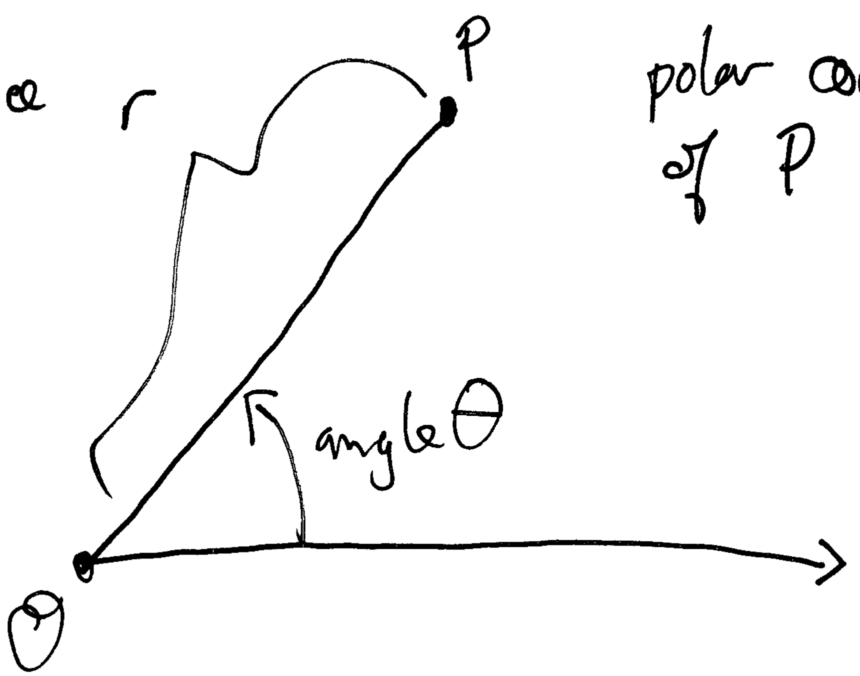
(Two distances)

Polar coordinates use a distance and an angle

O - origin or "pole"

ray from O = polar axis

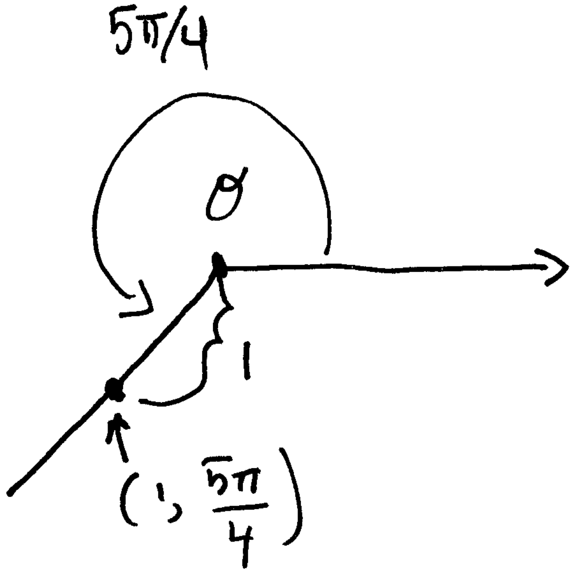
distance r

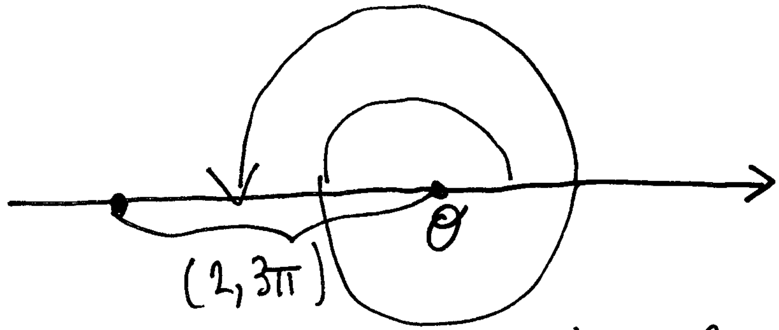



polar coordinates of P are (r, θ)

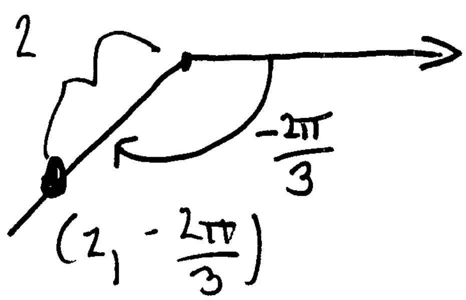
polar axis

Plot $\left(1, \frac{5\pi}{4}\right)$ $\cancel{(A, 3\pi)}$ $(2, 3\pi)$ $\left(2, -\frac{2\pi}{3}\right)$ $\left(-3, \frac{3\pi}{4}\right)$

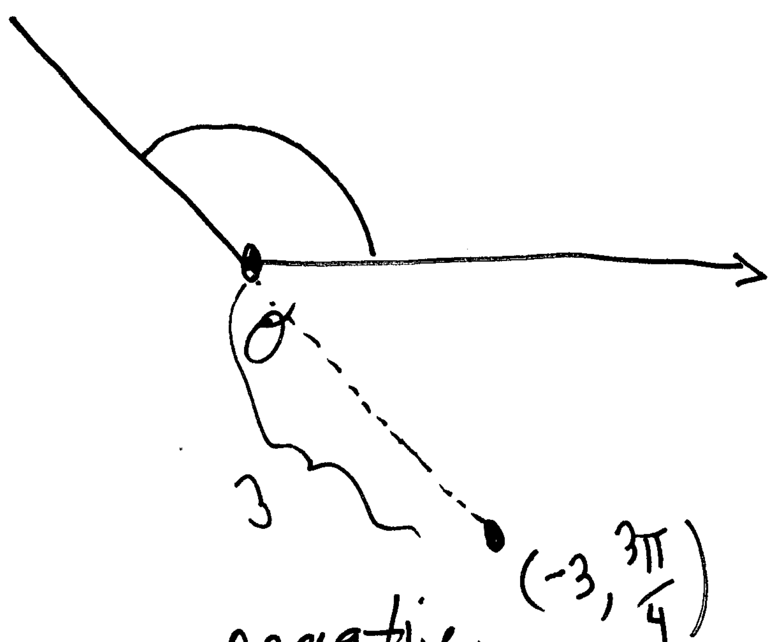



Angles bigger than $2\pi$ wrap around.




Negative angle




negative radius

Spelling out the conventions

* For any $r$ and $\theta$ the coordinates pairs

$$\left.\begin{array}{c} \vdots \\ (r, \theta - 4\pi) \\ (r, \theta - 2\pi) \\ (r, \theta) \\ (r, \theta + 2\pi) \\ (r, \theta + 4\pi) \\ \vdots \end{array}\right\} \quad (r, \theta + 2\pi n) \text{ for any } n$$

all represent the same point, geometrically.

* positive $\theta$ means <u>counterclockwise</u> rotation
  negative $\theta$ means <u>clockwise</u> rotation

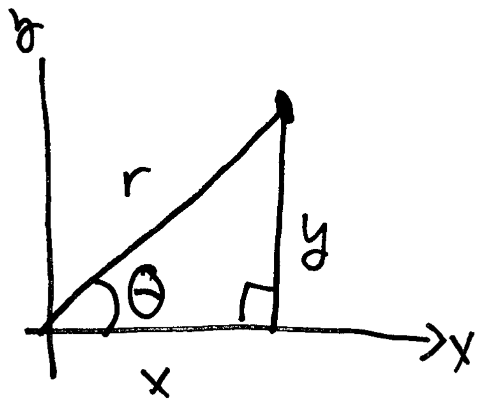
* negative $r$ means go in the negative direction along the line of angle $\theta$

  NOTE: $(-r, \theta)$ $(r, \theta + \pi)$

  represent the same point.

* $(0, \theta)$ represents the origin no matter what $\theta$ is.

## Convert from polar to cartesian




$$x = r\cos\theta$$
$$y = r\sin\theta$$

## Convert from cartesian to polar

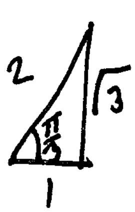
$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

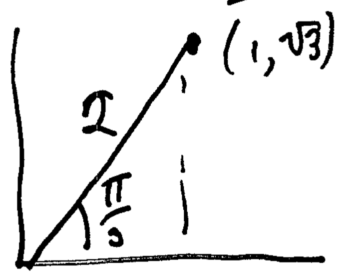
$$r = \pm\sqrt{x^2+y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) + \text{some multiple of } \pi$$

Since $r, \theta$ are not uniquely determined, need to check that the result lives in the correct quadrant.

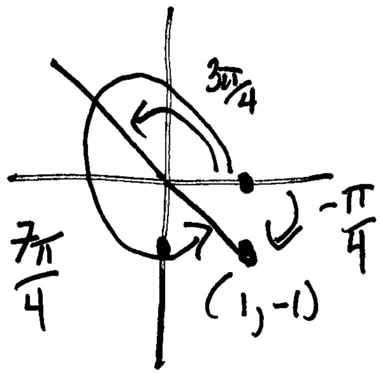
Ex if $(r,\theta) = (2, \pi/3)$

$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2\,\frac{1}{2} = 1$$
$$y = r\sin\theta = 2\sin\frac{\pi}{3} = 2\,\frac{\sqrt{3}}{2} = \sqrt{3}$$



if $(x,y) = (1,-1)$, what is $(r,\theta)$?




$$r^2 = x^2 + y^2 = 1^2 + (-1)^2 = 2$$

choose $r = \sqrt{2}$ (positive)

Now $\tan\theta = \frac{-1}{1} = -1$

$$\theta = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4}$$

$\frac{3\pi}{4}$ is wrong use $\frac{7\pi}{4}$

Answer $\left(\sqrt{2}, \frac{7\pi}{4}\right)$ or $\left(\sqrt{2}, \frac{-\pi}{4}\right)$ } both represent the same point.

can check

$$x = \sqrt{2}\cos\frac{-\pi}{4} = \sqrt{2}\frac{1}{\sqrt{2}} = 1$$

$$y = \sqrt{2}\sin\frac{-\pi}{4} = -\sqrt{2}\frac{1}{\sqrt{2}} = -1$$

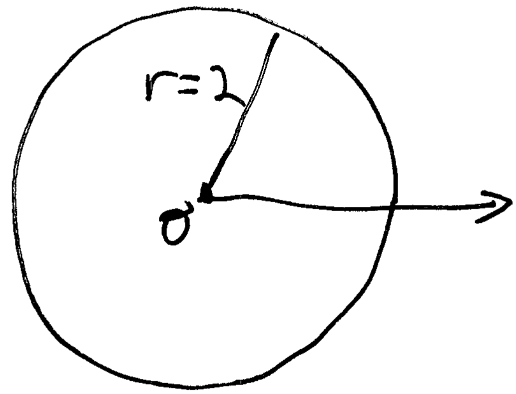
✓

Note $\left(-\sqrt{2}, \frac{3\pi}{4}\right)$ is also correct.

Graphs in polar coordinates

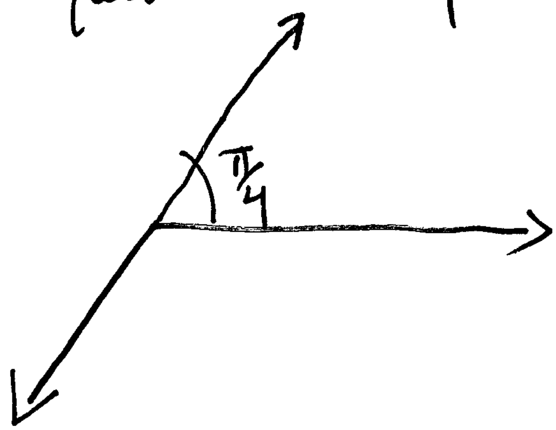
$r = f(\theta)$ or $F(r, \theta) = 0$

Polar equation $r = 2$




circle of radius 2

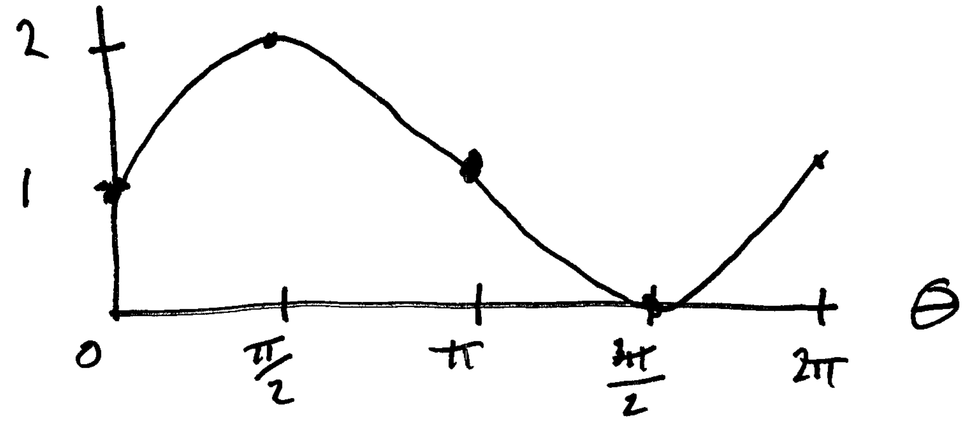
Polar equation $\theta = \frac{\pi}{4}$

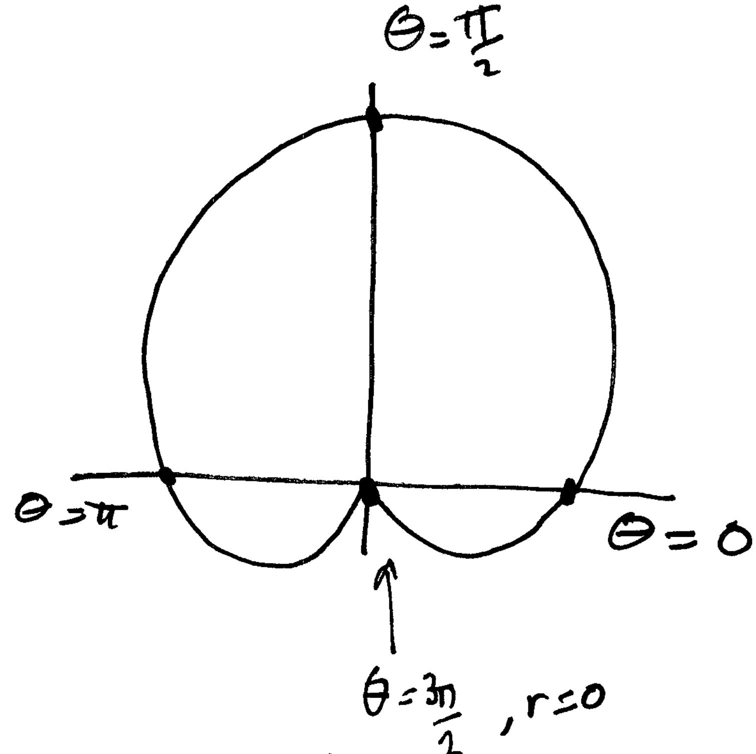



line which makes angle $\frac{\pi}{4}$ with the axis.

Polar equation $r = 1 + \sin\theta$

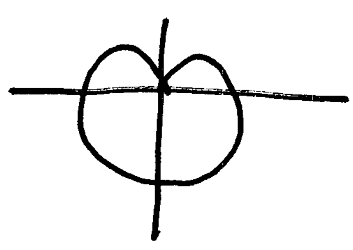
graph the right hand side $1+\sin\theta$




shape is called a "cardioid" "heart shape"

$r = (1 - \sin\theta)$



An equation $r = f(\theta)$ can be rewritten as a parametric equation in Cartesian coordinates:

use $\theta$ as the parameter

$$x = r\cos\theta = f(\theta)\cos\theta$$

$$y = r\sin\theta = f(\theta)\sin\theta$$

ex: $r = 1 - \sin\theta$

$$x = (1-\sin\theta)\cos\theta$$

$$y = (1-\sin\theta)\sin\theta$$

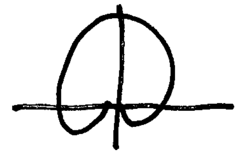
Tangent to a polar curve $\quad \dfrac{dy}{dx} = \dfrac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$

$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$$

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta$$

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} \qquad \left(\frac{dr}{d\theta} = f'(\theta)\right)$$

Ex Cardioid $r = 1 + \sin\theta$



(a) find slope of tangent line when $\theta = \pi/3$

(b) find horizontal & vertical tangents.

$$\frac{dr}{d\theta} = \cos\theta$$

$$\frac{dy}{dx} = \frac{(\cos\theta)\sin\theta + (1+\sin\theta)\cos\theta}{(\cos\theta)\cos\theta - (1+\sin\theta)\sin\theta}$$

$$= \frac{\cos\theta\,(1+2\sin\theta)}{\cos^2\theta - \sin^2\theta - \sin\theta}$$

(a) $$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{\left(\frac{1}{2}\right)\left(1+2\frac{\sqrt{3}}{2}\right)}{\left(\left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2 - \frac{\sqrt{3}}{2}\right)} = -1$$

Horizontal tangents

when $\frac{dy}{d\theta} = 0 = \cos\theta\,(1+2\sin\theta)$

$\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

or $1+2\sin\theta = 0 \Rightarrow \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

Vertical tangents

$$\frac{dx}{d\theta} = 0 = \cos^2\theta - \sin^2\theta - \sin\theta$$
$$= 1 - 2\sin^2\theta - \sin\theta$$
$$= (1+\sin\theta)(1-2\sin\theta)$$

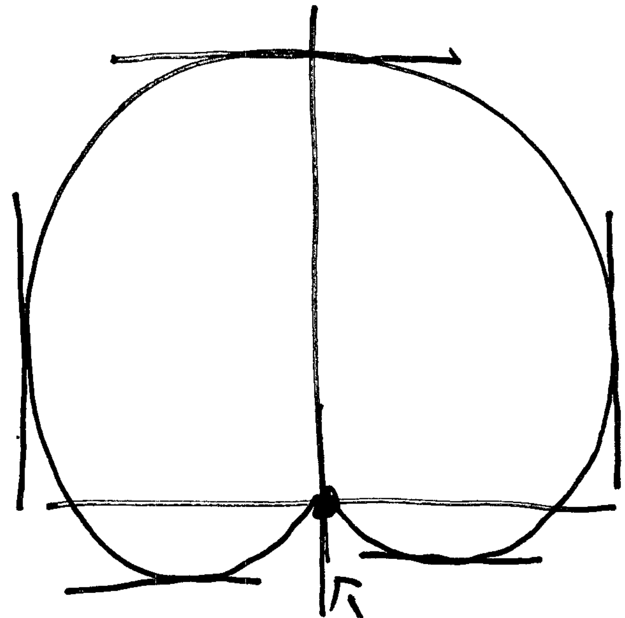
$\sin\theta = -1 \Rightarrow \theta = +\frac{3\pi}{2}$

or $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

At $\theta = \frac{3\pi}{2}$ both $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$

At $\theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$ Horizontal tangent

At $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ vertical tangent



$\theta = \frac{3\pi}{2}$ $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ both $0$.

at this point the <u>limiting tangent</u> is vertical