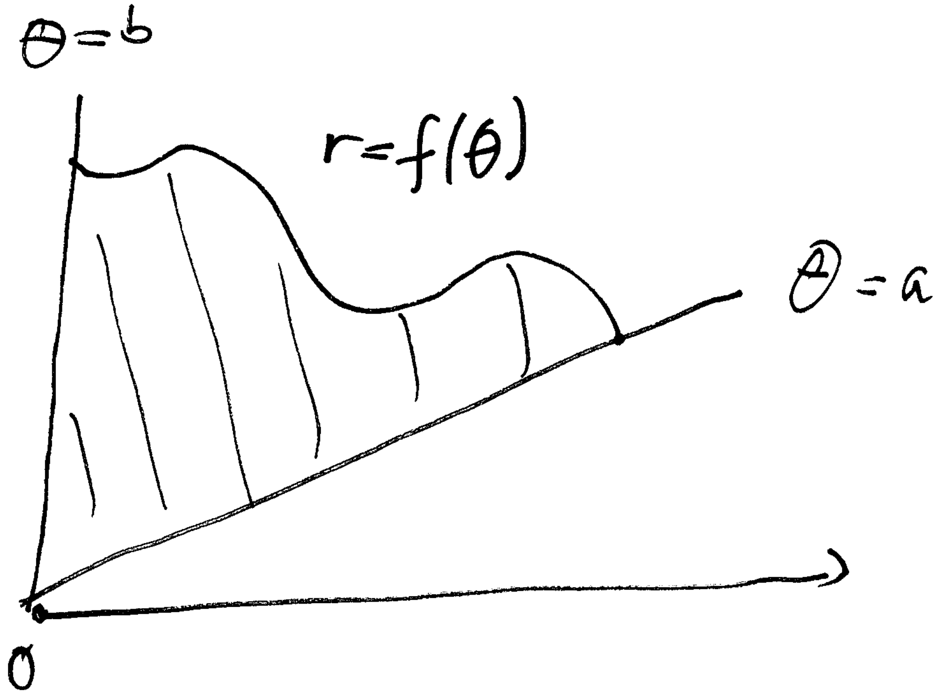
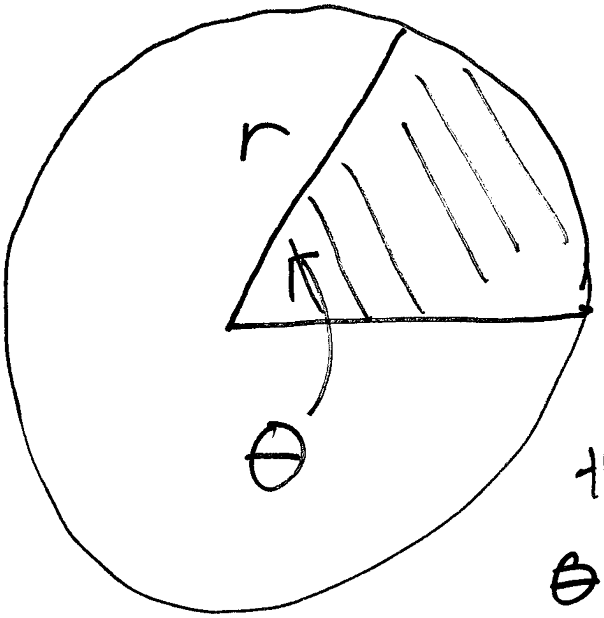
Areas & lengths in polar coordinates

Suppose we want to to find the area bounded by the curve $r = f(\theta)$ between the two lines $\theta = a$ and $\theta = b$



Since we're using polar coordinates, it's easiest to break the region up into sectors (wedge of a circle) rather than rectangles.

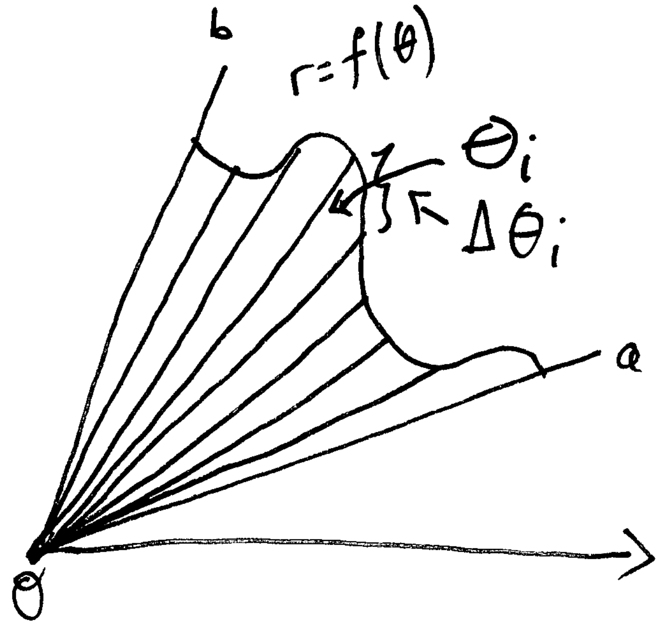


$\text{Area} = \frac{1}{2} r^2 \theta$

why? Area of total circle $= \pi r^2$

the sector represents a fraction $\frac{\theta}{2\pi}$ of the circle

$$A = \frac{\theta}{2\pi} \pi r^2 = \frac{1}{2} r^2 \theta$$



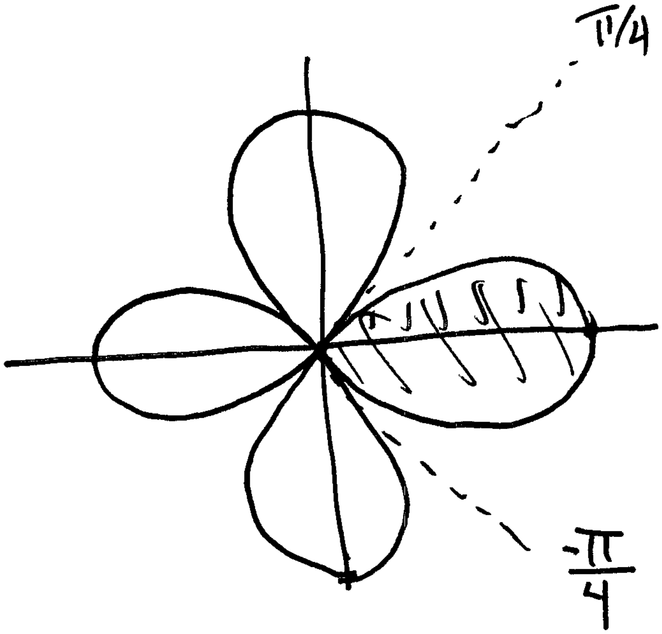
Area of ith sector $= \frac{1}{2} f(\theta_i)^2 \Delta\theta_i$

Total area $\approx \sum_i \frac{1}{2} f(\theta_i)^2 \Delta\theta_i$

Take limit as all $\Delta\theta_i \to 0$, sum becomes integral

$$A = \int_a^b \frac{1}{2} f(\theta)^2 \, d\theta$$

Example 1 Find the area enclosed by one loop of the "four leaf rose" $r = \cos 2\theta$



$$A = \int \frac{1}{2} r^2 \, d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \frac{1}{2} \cos^2 2\theta \, d\theta$$

$$= \frac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2 2\theta \, d\theta = \frac{1}{2} \, 2 \int_0^{\pi/4} \cos^2 2\theta \, d\theta$$

$$= \int_0^{\pi/4} \cos^2 2\theta \, d\theta = \int_0^{\pi/4} \frac{1}{2} (1 + \cos 4\theta) \, d\theta$$

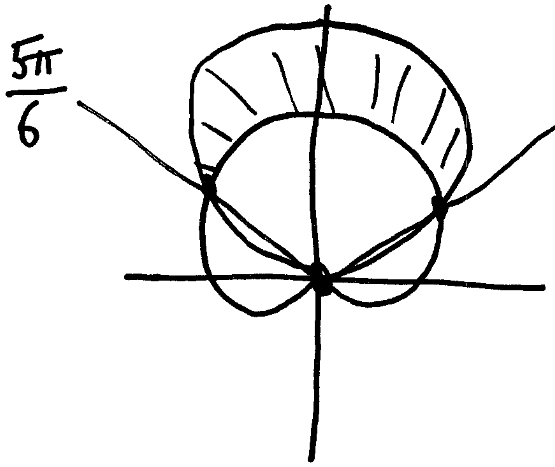
$$= \frac{1}{2} \left[\theta + \frac{1}{4} \sin 4\theta \right]_0^{\pi/4} = \frac{\pi}{8}$$

$$A = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}(3\sin\theta)^2 \, d\theta - \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}(1+\sin\theta)^2 \, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}\left[(3\sin\theta)^2 - (1+\sin\theta)^2\right] d\theta$$

In general area between $r = f(\theta)$ and $r = g(\theta)$ (assuming $f > g$)

$$A = \int_a^b \frac{1}{2}\left[f(\theta)^2 - g(\theta)^2\right] d\theta$$



$r = 3\sin\theta$ (circle)

$r = 1 + \sin\theta$ (cardioid)

$r = 3\sin\theta$ is a circle

$r = 3\sin\theta = 3\dfrac{y}{r}$ $\qquad r^2 = 3y$

$x^2 + y^2 = 3y$

center $\left(0, \frac{3}{2}\right)$

radius $= \frac{3}{2}$ $\qquad x^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{3}{2}\right)^2$

Tricky part limits of integration:
Need to find intersections in polar coordinates

$$3\sin\theta = r = 1 + \sin\theta$$

$$2\sin\theta = 1$$

$$\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

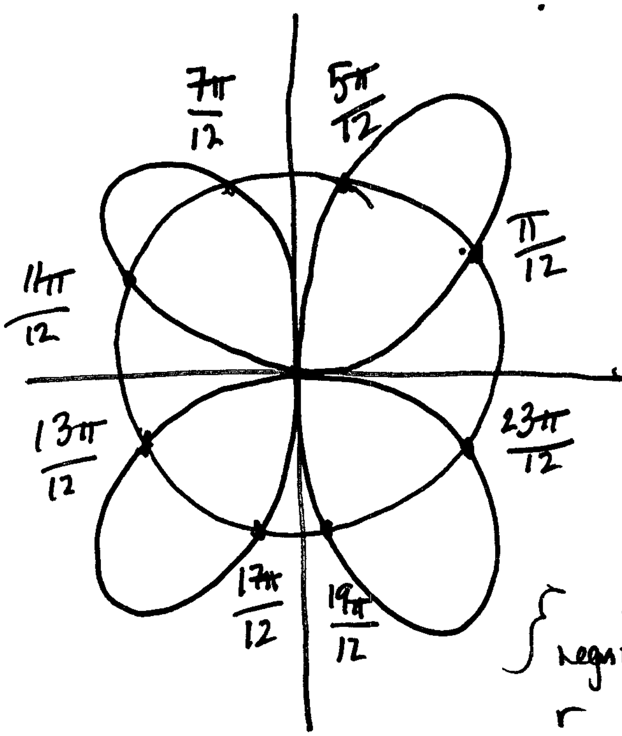
$(0, \theta)$ = origin for any θ

(r, θ) "=" $(-r, \theta + \pi)$

$r = 3\sin\theta$ passes thru O when $\theta = 0$

$r = 1 + \sin\theta$ passes thru O when $\theta = \frac{3\pi}{2}$

$r = 2\sin 2\theta$, $r = 1$ Find all intersections



negative r

Definitely 8 intersections

solve $1 = 2\sin 2\theta$

$\frac{1}{2} = \sin 2\theta$

$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \ldots$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

also solve $-1 = 2\sin 2\theta$

Arc length

$$x = r\cos\theta = f(\theta)\cos\theta$$

$$y = r\sin\theta = f(\theta)\sin\theta$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$\left.\begin{aligned} \frac{dx}{d\theta} &= \frac{dr}{d\theta}\cos\theta - r\sin\theta \\ \frac{dy}{d\theta} &= \frac{dr}{d\theta}\sin\theta + r\cos\theta \end{aligned}\right\} \text{did this last time for tangents}$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \cdots \overset{\text{trig}}{=} \left(\frac{dr}{d\theta}\right)^2 + r^2$$

$$L = \int_a^b \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta$$

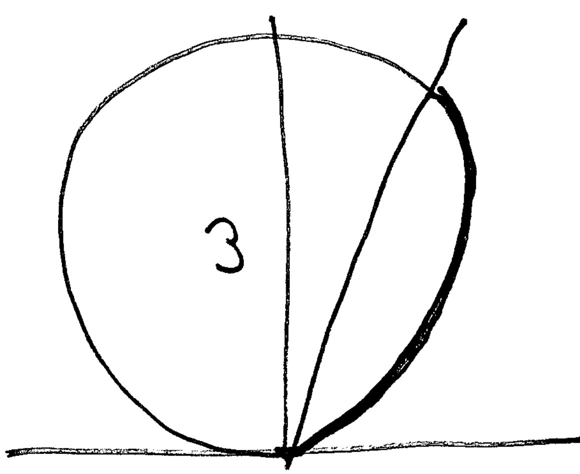
Find length $r = 3\sin\theta \quad (0 \le \theta \le \frac{\pi}{3})$

$$\frac{dr}{d\theta} = 3\cos\theta$$

$$L = \int_0^{\pi/3} \sqrt{(3\cos\theta)^2 + (3\sin\theta)^2}\, d\theta$$

$$= \int_0^{\pi/3} \sqrt{9(\cos^2\theta + \sin^2\theta)}\, d\theta$$

$$= 3\int_0^{\pi/3} d\theta = 3\frac{\pi}{3} = \pi$$



diam = 3

circumference = 3π

integral above

$= \frac{1}{3}$ (total circumference)