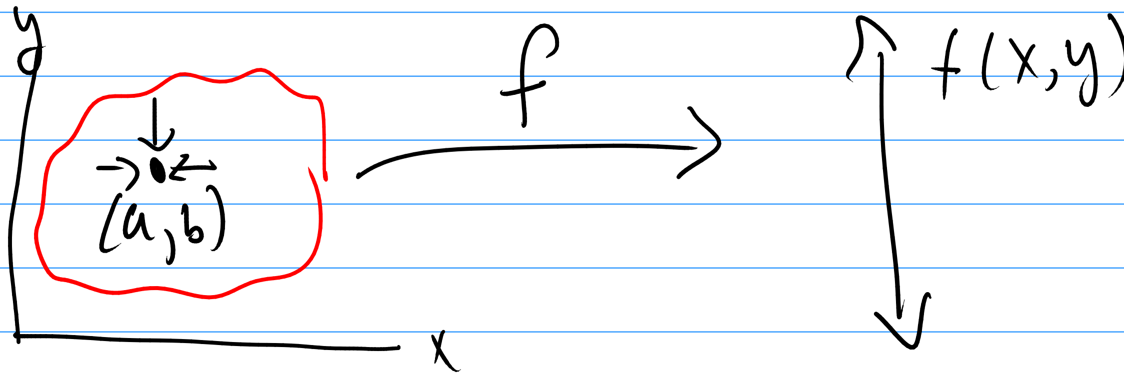
Partial Derivatives

$f(x,y)$ - function of 2 variables



Recall: $\lim_{(x,y)\to(a,b)} f(x,y) = L$

means that $f(x,y)$ approaches L

as (x,y) approaches (a,b) from any direction

Note: we can take limits in any direction.

Today: We can take derivatives

in different directions

$f_x = \frac{\partial f}{\partial x}$ partial derivative with respect to x

$f_y = \frac{\partial f}{\partial y}$ partial derivative with respect to y

To compute $f_x(a,b)$,

Define a new function $g(x) = f(x,b)$

Now take $g'(x)$, and plug in a:

$$f_x(a,b) = g'(a)$$

"Differentiate f with respect to x, while holding y constant"

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h,b) - f(x,b)}{h} = f_x(x,b)$$

$$f_y(a,y) = \lim_{h \to 0} \frac{f(a,y+h) - f(a,y)}{h}$$

In general, f_x and f_y are completely different functions

$$f(x,y) = e^{-y} \cos \pi x$$

$f_x \rightarrow$ hold y constant, so e^{-y} is also constant
differentiate with respect to x

$$f_x = e^{-y}(-\pi \sin \pi x)$$

$$f_y = (-e^{-y}) \cos \pi x$$

<u>Ex</u> $\quad f(x,y) = x^2 + y^2$

$f_x = 2x + 0 \qquad f_y = 0 + 2y$

NOTATIONS $z = f(x, y)$

partial derivative w.r.t. x

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

Partial derivatives w.r.t. y

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

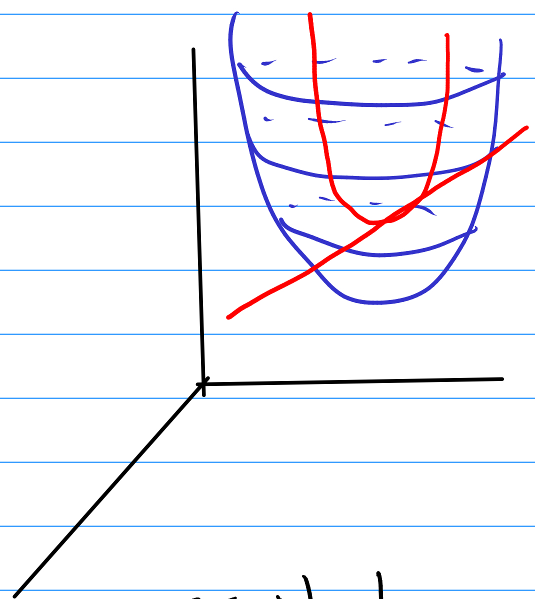
New symbol

∂ curly d denotes partial, as opposed to total, derivative

A partial derivative is just like an ordinary derivative, where all variables but one are held constant.

Geometric Interpretation

Graph $z = f(x,y)$



$f_x(a,b)$

slice with the plane $y=b$, then take slope of the intersection at (a,b)

Similarly for $f_y(a,b)$, slice with a plane $x=a$

Ex $f(x,y) = x^2 + 2y^2 - 4$

what are $f_x(1,1)$ and $f_y(1,1)$?

$f_x(x,y) = 2x \qquad f_x(1,1) = 2(1) = 2$

$f_y(x,y) = 4y \qquad f_y(1,1) = 4$

More than two variables

* One variable at a time!

(Hold all but one variable constant

$f(x,y,z) \rightarrow f_x(x,y,z)$ hold y,z constant

$\rightarrow f_y(x,y,z)$ hold x,z constant

$\rightarrow f_z(x,y,z)$ hold x,y constant

$$u = f(x_1, x_2, \dots, x_n)$$

$$u_{x_i} = \lim_{h \to 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h}$$

Ex $f(x,y,z) = e^{xy} \ln z$

$$f_x = \frac{\partial}{\partial x}\left(e^{xy} \ln z\right) = \frac{\partial}{\partial x}\left(e^{xy}\right) \ln z$$

$$= e^{xy} \frac{\partial}{\partial x}(xy) \ln z = y e^{xy} \ln z$$

$f(x,y)$

Higher derivatives $\quad f_{xx}$

$f(x,y) \to f_x = \frac{\partial f}{\partial x} \to f_{xx} = \frac{\partial}{\partial x}(f_x)$

function of 2 vars $\qquad$ function of 2 vars

$$f_{xx} = \frac{\partial}{\partial x}\frac{\partial}{\partial x} f = \frac{\partial^2}{\partial x^2} f$$

$$f_{yy} = \frac{\partial}{\partial y}\frac{\partial}{\partial y} f = \frac{\partial^2}{\partial y^2} f$$

$$f_{yx} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f = \frac{\partial^2}{\partial x \partial y} f$$

$$f_{xy} = \frac{\partial}{\partial y}\frac{\partial}{\partial x} f = \frac{\partial^2}{\partial y \partial x} f$$

Clairaut's theorem: $f_{xy} = f_{yx}$

provided that both of these functions are continuous.

"Equality of mixed partials"

$f_{xyy} = \frac{\partial}{\partial y}(f_{xy})$ etc.

__Ex__ $f(x,y) = x^3y^5 + 2x^4y$

$f_x = 3x^2y^5 + 8x^3y$ $\quad$ $f_y = x^3(5y^4) + 2x^4$

$f_{xx} = 6xy^5 + 24x^2y$ $\quad$ $f_{yy} = x^3(20y^3) + 0$

$f_{xy} = 3x^2(5y^4) + 8x^3$ $\quad$ $f_{yx} = 3x^2(5y^4) + 8x^3$

These two are equal!