Parametric Equations

describe curves in the xy-plane not as $y = f(x)$, but by writing x and y as functions of yet another variable t.

t - parameter "time"

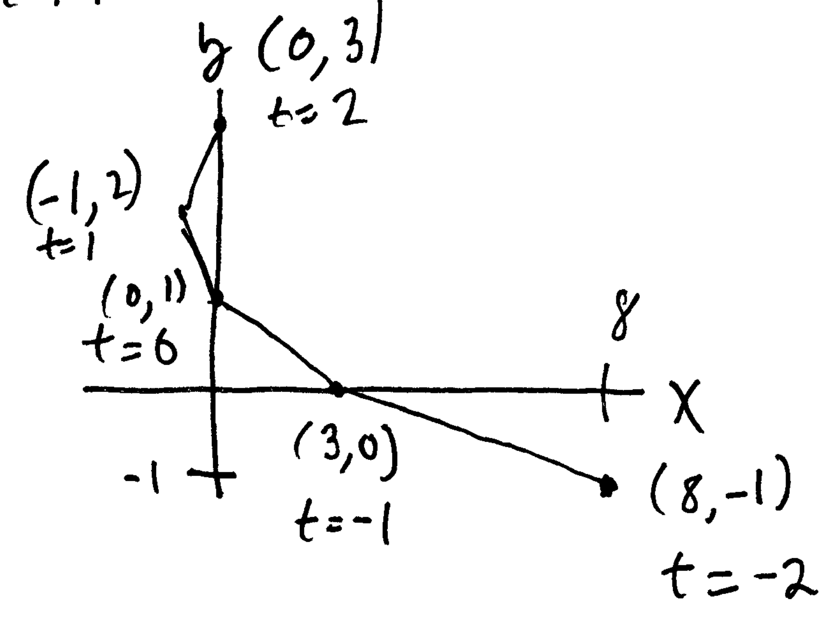
$$\begin{cases} x = f(t) \\ y = g(t) \end{cases} \qquad (x,y) = (f(t), g(t))$$

$$(x,y) = (x(t), y(t))$$

<u>Example</u>

$$\begin{cases} x(t) = t^2 - 2t \\ y(t) = t + 1 \end{cases}$$

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3



As t runs through different values, $(x(t), y(t))$ traces out a curve in the plane.

In this case, the curve is sideways parabola.

We can "eliminate the parameter" to get the "Cartesian equation" of the curve.

$$x = t^2 - 2t$$

$$y = t + 1 \Rightarrow t = y - 1$$

$$x = (y-1)^2 - 2(y-1)$$ "eliminating the parameter"

$$x = y^2 - 2y + 1 - 2y + 2 = y^2 - 4y + 3$$

$$x = y^2 - 4y + 3$$ equation for Parabola

The Cartesian equation contains <u>less information</u> than the parametric equations:

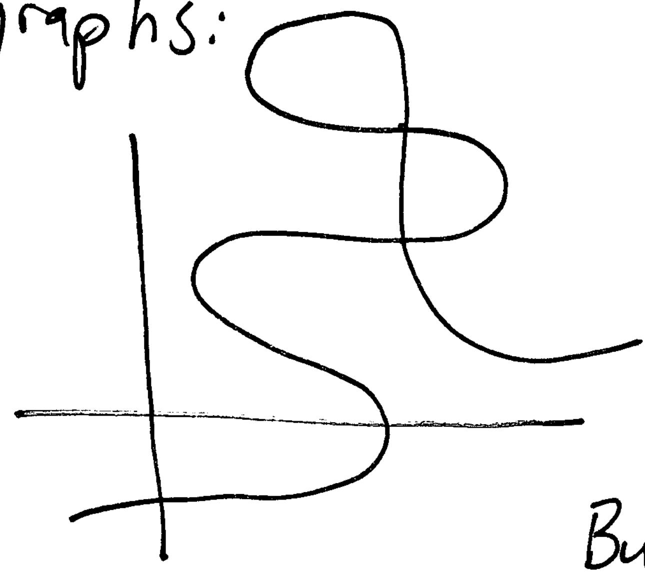
The parametric equations attach a "time" / value of t to each point on the curve.

They tell us "when" the "particle" was at a particular location.

At time t, at $(x(t), y(t))$

In physics, they describe the trajectory of a particle.

<u>Aside</u>: parametric equations let you write formulas for curves that are not graphs:



Cannot be the graph of a function

But it does have a parametric description.

self-intersection

$\iff$ two values of t where (x, y) are equal

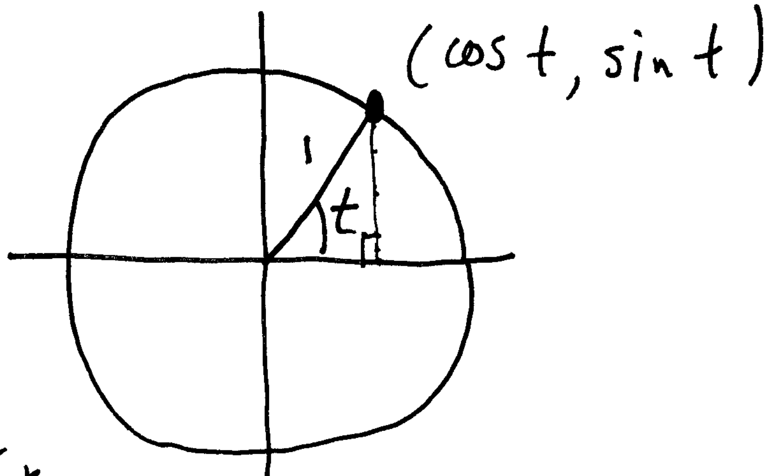
t_1, t_2 such that $\begin{cases} x(t_1) = x(t_2) \\ y(t_1) = y(t_2) \end{cases}$

Example everyone should know:

Parametric equations for a circle
a.k.a. "Uniform circular motion"

$$\begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases} \quad \text{for } 0 \le t \le 2\pi$$

since
$(\cos t, \sin t)$
are the coordinates
of a point on
the circle
these parametric
equations trace out a circle.

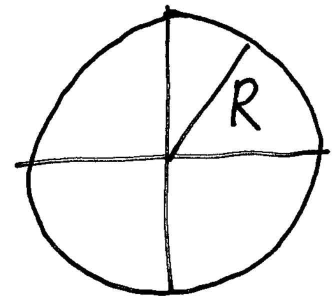


Pythagorean identity $\cos^2 t + \sin^2 t = 1$

Cartesian equation
for a circle $\rightarrow x^2 + y^2 = 1$

What if we want the Radius to be R?

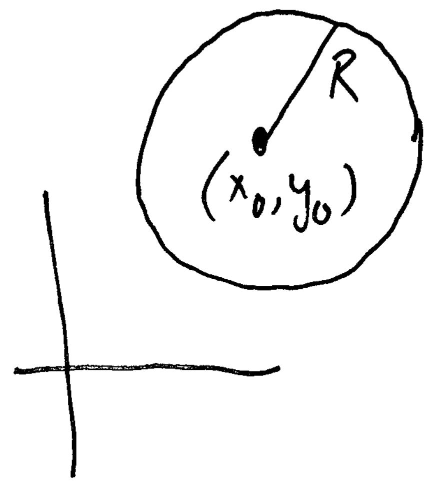
$$\begin{cases} x(t) = R\cos t \\ y(t) = R\sin t \end{cases}$$



$$x^2 + y^2 = R^2$$

What if I want the center to be at (x_0, y_0)

just add it in:



$$\begin{cases} x(t) = R\cos t + x_0 \\ y(t) = R\sin t + y_0 \end{cases}$$

$$(x - x_0)^2 + (y - y_0)^2 = R^2$$

One more thing to fiddle with: ~~pa~~

"The parameterization".

Can parameterize the same curve in different ways

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}$$

As t goes from 0 to 2π, these parametric equations trace out the circle once.

What if I want the circle to be traced once as t goes from 0 to T?

Replace t by $\frac{2\pi}{T} t$.

$$\begin{cases} x = \cos\left(\frac{2\pi}{T} t\right) \\ y = \sin\left(\frac{2\pi}{T} t\right) \end{cases} \qquad 0 \le t \le T$$

T = period

$1/T$ = frequency

$2\pi/T$ = angular frequency

<u>Cycloid</u> = curve traced out by a point on the rim of a wheel as the wheel rolls.

initial configuration.

Parameter = θ = the angle through which the wheel has rotated since the initial configuration.

<u>Step 1</u>: figure out how the center of the wheel moves.

After rotation through angle θ, the wheel rolls to the right a distance $r\theta$.

distance $= r\theta$

$$\begin{cases} x_{center} = r\theta \\ y_{center} = r \end{cases}$$

Step 2 How does the point on the rim move relative to the center?

It rotates clockwise from its initial position at the bottom.

$r\cos\left(-\frac{\pi}{2} - \theta\right)$

$r\sin\left(-\frac{\pi}{2} - \theta\right)$

why $-\frac{\pi}{2} - \theta$?
starts at the bottom
$\Rightarrow$ when $\theta = 0$,
P makes an angle $-\frac{\pi}{2}$
with the horizontal
rotates clockwise add $-\theta$.

Add them

$$\begin{cases} x(t) = r\cos\left(-\frac{\pi}{2} - \theta\right) + r\theta \\ y(t) = r\sin\left(-\frac{\pi}{2} - \theta\right) + r \end{cases}$$

$$\begin{cases} x(t) = r(-\sin\theta) + r\theta = r(\theta - \sin\theta) \\ y(t) = r(-\cos\theta) + r = r(1 - \cos\theta) \end{cases}$$