Max/Min

<u>Recall</u> $f(x,y,z)$

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} = \langle f_x, f_y, f_z \rangle$$

<u>gradient</u>

Chain Rule and gradient

$\vec{r}(t)$ curve $\quad \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$f(\vec{r}(t))$ real-valued function of t

$$\frac{d}{dt} f(\vec{r}(t)) = \frac{d}{dt} f(x(t), y(t), z(t))$$

$$= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

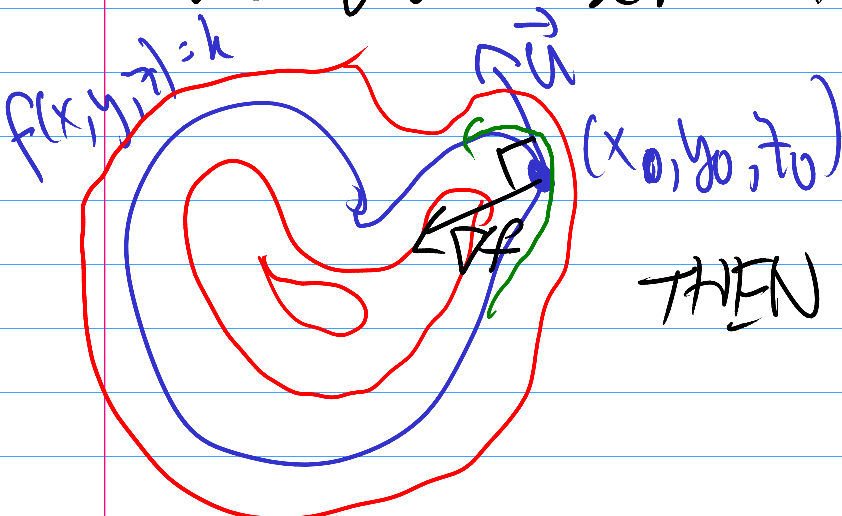
$$= \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) = \frac{d}{dt} f(\vec{r}(t))$$

(gradient $\cdot$ velocity)

Level set $f(x,y,z)=k$

<u>Theorem</u> suppose (x_0,y_0,z_0) in the level set $f(x_0,y_0,z_0)=k$

Suppose $\vec{u}$ is a tangent vector to the level set at (x_0,y_0,z_0)



THEN $\nabla f(x_0,y_0,z_0)\cdot\vec{u}=0$

<u>Proof</u> let $\vec{r}(t)$ be a curve which has

$$\vec{r}(0)=(x_0,y_0,z_0)$$
$$\vec{r}\,'(0)=\vec{u}$$

and $f(\vec{r}(t))=k$ for all t

($\vec{r}(t)$ is contained in the level set)

$$\frac{d}{dt}f(\vec{r}(t))\Big|_{t=0}=\nabla f(\vec{r}(0))\cdot\vec{r}\,'(0)$$
$$=\nabla f(x_0,y_0,z_0)\cdot\vec{u}$$

on the other hand $f(\vec{r}(t)) = k$

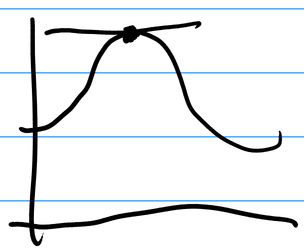
$$\frac{d}{dt} f(\vec{r}(t))\Big|_{t=0} = \frac{d}{dt}[k]\Big|_{t=0} = 0$$

QED

Max / min for functions of 2-variables

Recall 1-variable:

$f(x)$: Every local max or min occurs at a point where $f'(x)=0$ CRITICAL POINT



Either $f''(x)=0$ degenerate critical point

Could be a max $-x^4$

min x^4

neither x^3

If $f''(x) \neq 0$ nondegenerate critical point

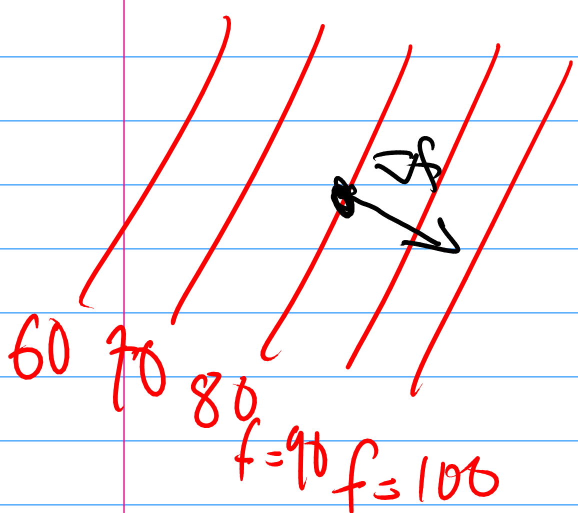
$f''(x) > 0$ LOCAL MIN $f(x)=x^2$

$f''(x) < 0$ LOCAL MAX $f(x)=-x^2$

(Also need to check endpoints of domain)

Critical point of $f(x,y)$ is where <u>both</u> partial derivatives vanish, that is, the gradient vanishes.

$$\left[\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \quad \text{or} \quad \nabla f = 0 \right] \text{CRITICAL POINT}$$



gradient points in a direction where the function increase

if $\nabla f \neq 0$, the point can't be a max/min

For second derivative test consider

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

Hessian matrix $\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = H$

$$D = \det H = f_{xx} f_{yy} - \underbrace{f_{xy} f_{yx}}_{(f_{xy})^2}$$

b/c $f_{xy} = f_{yx}$

* $f(x,y) = x^2 + y^2$ MIN

$\nabla f = \langle 2x, 2y \rangle$

$H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $D = 4 > 0$

$f_{xx} = 2 > 0$

* $f(x,y) = -x^2 - y^2$ MAX

$\nabla f = \langle -2x, -2y \rangle$

$H = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ $D = 4 > 0$

$f_{xx} = -2 < 0$

* $f(x,y) = x^2 - y^2$ SADDLE

$\nabla f = \langle 2x, -2y \rangle$

$H = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ $D = -4$

FACT: Every critical point with $D \neq 0$ looks like one of these examples

$D = 0$ degenerate critical point
second derivs don't tell you anything

$D \neq 0$ nondegenerate critical point

$D > 0$ max OR min

$f_{xx} > 0$ min
$f_{xx} < 0$ max

can use f_{yy} instead

$D < 0$ saddle

Ex Classify the critical points of $f(x,y) = x^3y + 12x^2 - 8y$

$$\frac{\partial f}{\partial x} = 3x^2y + 24x = 0$$

$$\frac{\partial f}{\partial y} = x^3 - 8 = 0 \Rightarrow x = 2$$

$$3(2)^2y + 24(2) = 0$$

$$12y + 48 = 0$$

$$y = -4$$

So $(2,-4)$ is the critical point.

$f_{xx} = 6xy + 24$ $\quad f_{yy} = 0$

$f_{xy} = 3x^2$

AT $(2,-4)$ $\quad f_{xx} = -24$ $\quad f_{yy} = 0$

$f_{xy} = 12$

$$D = f_{xx}f_{yy} - (f_{xy})^2 = (-24)(0) - (12)^2$$

$$= -144 < 0$$

SADDLE //