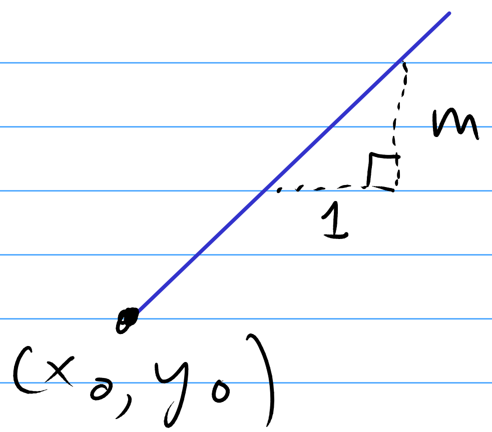
Lines and Planes in terms of vectors

Lines: in 2 dimensions, a line is determined by a point (x_0, y_0) and a slope m:

$$(y - y_0) = m(x - x_0)$$



The number m controls the direction of the line.

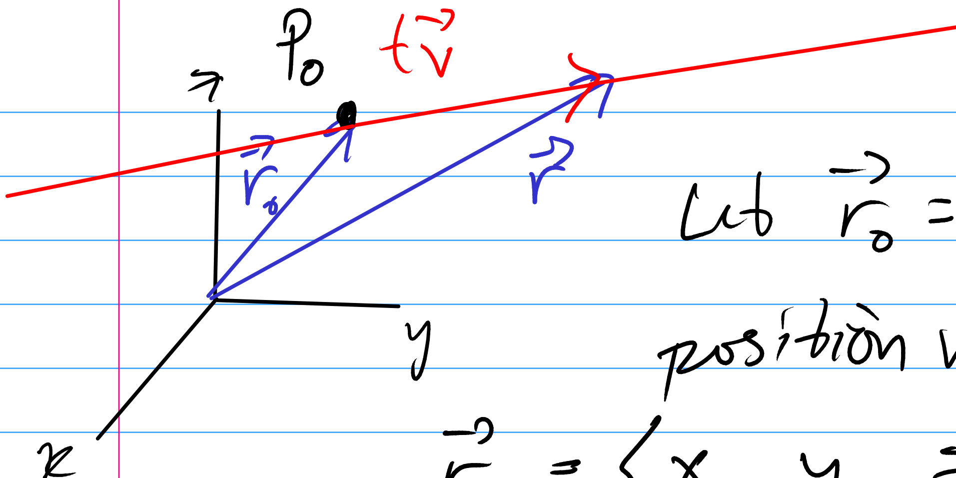
Generalization to 3 dimensions

A line is determined by

a point $P_0(x_0, y_0, z_0)$ and

a direction, that is, a vector

$$\vec{v} = \langle a, b, c \rangle$$



Let $\vec{r}_0 = \vec{OP_0}$ be the position vector of P_0.

$$\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$$

(vector whose components are the coordinates of P_0)

Let the position vector of another point on the line be $\vec{r}$. Then

$\vec{r} - \vec{r}_0 = t\vec{v}$ is a multiple of the direction vector $\vec{v}$.

Thus $\vec{r} = \vec{r}_0 + t\vec{v}$ is the vector equation of the line.

It gives the position vector $\vec{r}$ of a point on the line in terms of the parameter t, from the data $\vec{r}_0$ and $\vec{v}$.

In components

$P_0 = (x_0, y_0, z_0)$ so $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$

$\vec{v} = \langle a, b, c \rangle$

$\vec{r} = \vec{r}_0 + t\vec{v} =$

$\langle x_0 + at, y_0 + bt, z_0 + ct \rangle$

This could also be written as a system of parametric equations

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

Or eliminate the parameter t if you like:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

[Note: this is a system of 2 equations]

$\underline{\text{Ex}}$: Find equations for line through $(5,1,3)$ in direction $\vec{v} = \vec{i} + 4\vec{j} - 2\vec{k}$

$$\vec{r}_0 = \langle 5, 1, 3 \rangle = 5\vec{i} + \vec{j} + 3\vec{k}$$

$$\vec{r} = \vec{r}_0 + t\vec{v} = 5\vec{i} + \vec{j} + 3\vec{k} + t(\vec{i} + 4\vec{j} - 2\vec{k})$$

$$= (5+t)\vec{i} + (1+4t)\vec{j} + (3-2t)k$$

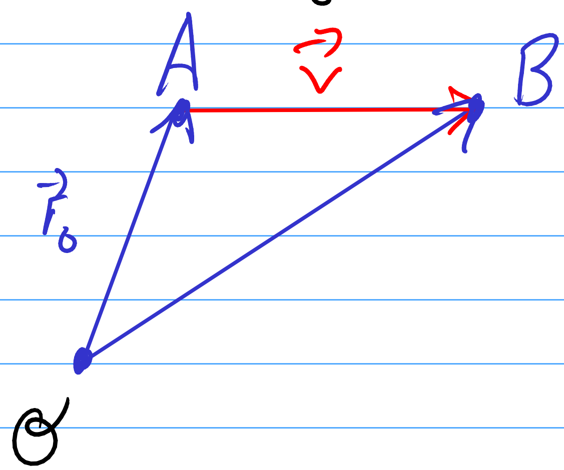
$$= \langle 5+t, 1+4t, 3-2t \rangle$$

OR $\begin{cases} x = 5+t \\ y = 1+4t \\ z = 3-2t \end{cases}$

OR $\dfrac{x-5}{1} = \dfrac{y-1}{4} = \dfrac{z-3}{-2}$

Also true that two points A B determine a line in 3 dimensions.

use $\vec{r}_0 = \overrightarrow{OA}$ $\quad \vec{v} = \overrightarrow{AB}$



<u>Ex</u> $A = (2, 4, -3) \quad B = (3, -1, 1)$

$$\vec{r}_0 = \overrightarrow{OA} = \langle 2, 4, -3 \rangle$$

$$\vec{v} = \overrightarrow{AB} = \langle 3-2, -1-4, 1-(-3) \rangle$$

$$= \langle 1, -5, 4 \rangle$$

Now as before

$$\vec{r} = \vec{r}_0 + t\vec{v} = \langle 2+t, 4-5t, -3+4t \rangle$$

Question: where does this line intersect the yz-plane?

yz-plane is $\{x=0\}$.

so set $x = 2+t = 0 \Rightarrow t = -2$

so $\vec{r} = \langle 2+(-2), 4-5(-2), -3+4(-2) \rangle$

$= \langle 0, 14, -11 \rangle$

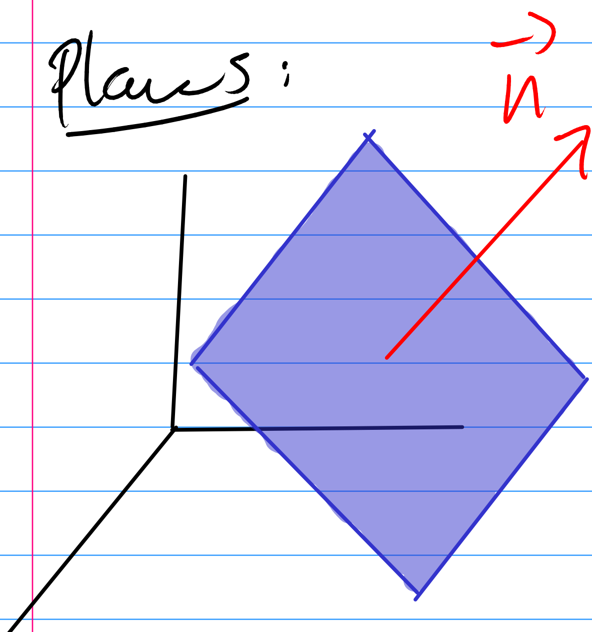
so they intersect at $(0, 14, -11)$. $\blacksquare$

NOTE: the equation for a particular line is not unique; we can use any point on the line and any vector parallel to the line to build the equation.

Two lines in 3d are parallel if their direction vectors are parallel.

BUT, in 3 dimensions, non-parallel lines need not intersect, such lines are called skew.

Planes:



The "direction" of a plane is determined by a vector perpendicular to the plane

This is the normal vector $\vec{n}$

(perpendicular = orthogonal = normal)

The idea is that the vector between two points in the plane is perpendicular to $\vec{n}$.

If $\vec{r}_0$ is the position vector of some fixed point in the plane, and $\vec{r}$ is the position vector of any other point in the plane, then

$\vec{n}$ is perp. to $(\vec{r} - \vec{r}_0)$

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0.$$

Or $\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$

In components: $\vec{n} = \langle a, b, c \rangle$

$\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$, $\vec{r} = \langle x, y, z \rangle$

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Ex plane through $(2,4,-1)$
normal to $\vec{n} = \langle 2,3,4 \rangle$

$$2(x-2) + 3(y-4) + 4(z-(-1)) = 0$$
$$2x - 4 + 3y - 12 + 4z + 4 = 0$$
$$2x + 3y + 4z = 12$$

Ex plane containing 3 points
$P = (1,3,2)$ $Q = (3,-1,6)$ $R = (5,2,0)$
contains vectors $\vec{PQ} = \langle 2,-4,4 \rangle$
$\vec{PR} = \langle 4,-1,-2 \rangle$

A normal vector is perpendicular to both $\vec{PQ}$ and $\vec{PR}$, so take cross product

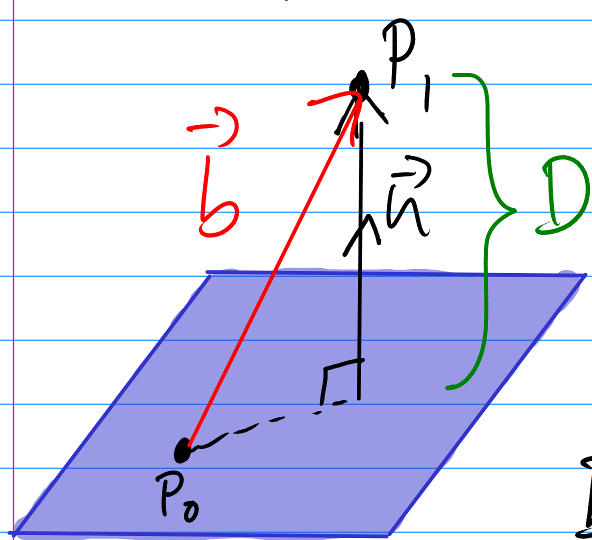
$$\vec{n} = \vec{PQ} \times \vec{PR} = \langle 12, 20, 14 \rangle$$

with $\vec{r}_0 = \overrightarrow{OP} = \langle 1,3,2 \rangle$ and thus $\vec{n}$,

the equation is

$$12(x-1) + 20(y-3) + 14(z-2) = 0$$

The point-to-plane distance formula is an application of projections.



Plane contains P_0, with normal vector $\vec{n}$

P_1 another point

$\vec{b} = \overrightarrow{P_0P_1}$ difference vector.

distance point to plane = D = length along line parallel to $\vec{n}$ from plane to P_1

$$= \left| \mathrm{comp}_{\vec{n}} \vec{b} \right| = \frac{|\vec{n} \cdot \vec{b}|}{|\vec{n}|}$$

$\vec{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$

$\vec{n} = \langle a, b, c \rangle$

$\vec{n} \cdot b = a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)$

$|\vec{n}| = \sqrt{a^2 + b^2 + c^2}$

So: $$D = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}.$$

Note that $a(x - x_0) + b(y - y_0) + c(z - z_0)$

$= ax + by + cz - d$

where $d = ax_0 + by_0 + cz_0$

So $$D = \frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

If the plane has equation

$ax + by + cz = d.$

added after lecture

Ex find distance between parallel planes

$10x + 2y - 2z = 5$ and $5x + y - z = 1$

First note that the planes are parallel because the normal vectors

$\langle 10, 2, -2 \rangle$ and $\langle 5, 1, -1 \rangle$

are parallel vectors

Distance between parallel planes

= Distance between one plane and any point on the other.

On the first plane set $y = z = 0$

$10x + 0 - 0 = 5 \Rightarrow x = \frac{1}{2}$

Thus $(\frac{1}{2}, 0, 0)$ is a point on the first plane

Distance formula

$\left(\frac{1}{2}, 0, 0\right)$ and $5x + y - z = 1$

$$\frac{\left|5\left(\frac{1}{2}\right) + 0 - 0 - 1\right|}{\sqrt{5^2 + 1^2 + (-1)^2}} = \frac{\left(3/2\right)}{\sqrt{27}}$$

$$= \frac{3}{2} \cdot \frac{1}{3\sqrt{3}} = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}.$$