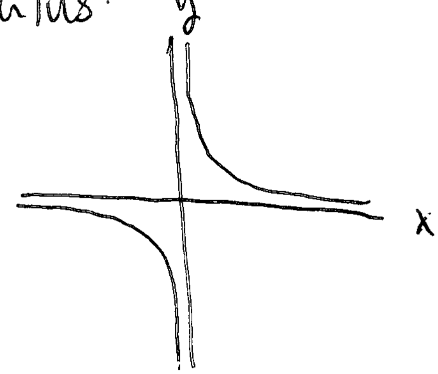
M408D Indeterminate forms & L'Hospital's Rule

Recall from first semester of Calculus:

$f(x) = \frac{1}{x}$ graph:



* Notice horizontal & vertical asymptotes

In terms of limits:

$\lim_{x \to \infty} \frac{1}{x} = 0$, $\lim_{x \to -\infty} \frac{1}{x} = 0$ " $\frac{1}{\pm\infty} = 0$ "

$\lim_{x \to 0^+} \frac{1}{x} = +\infty$, $\lim_{x \to 0^-} \frac{1}{x} = -\infty$ " $\frac{1}{0} = \pm\infty$ "

* These limits tell us what to do when we see $\frac{1}{0}$ or $\frac{1}{\infty}$ in a limit.

sign must be checked, and may need to take one-sided limit

* Question: what to do when we see $\frac{0}{0}$ or $\frac{\infty}{\infty}$?

For that matter, what about other undefined expressions such as $0 \cdot \infty$, $\infty - \infty$, 0^0, ∞^0, 1^∞, ...?

(These are called <u>indeterminate forms</u>)

More precisely, suppose $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$.

Q. What is $\lim_{x \to a} \frac{f(x)}{g(x)}$?

A: We <u>don't know</u> without further information:

* If $f(x)$ goes to 0 "faster" than $g(x)$, the quotient goes to 0.
* If $g(x)$ goes to 0 "faster" than $f(x)$, the quotient goes to $\pm\infty$.
* If $f(x)$ and $g(x)$ go to 0 "at the same rate" then the limit may stabilize at a finite nonzero number.

(The same kind of story is true for all the indeterminate forms.)

TOOL: L'Hospital's Rule:

If you see $\frac{0}{0}$ or $\frac{\infty}{\infty}$, you can differentiate the numerator and denominator, and you may get an easier problem:

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ OR $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$

$$\boxed{\text{Then } \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

(If this limit exists)

Example $\lim_{x \to 0} \frac{e^x - 1}{x}$

Try to plug in: $\frac{e^0 - 1}{0} = \frac{1-1}{0} = \frac{0}{0}$

So, we can use L'Hospital's Rule

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{\frac{d}{dx}(e^x - 1)}{\frac{d}{dx}(x)} = \lim_{x \to 0} \frac{e^x}{1} = \frac{e^0}{1} = 1$$

Example $\lim_{x \to \pi} \frac{\sin x}{1 - \cos x}$

Plug in: $\lim_{x \to \pi} \frac{\sin x}{1 - \cos x} = \frac{\sin \pi}{1 - \cos \pi} = \frac{0}{1-(-1)} = \frac{0}{2} = 0.$

We're done! No Need to use L'Hospital's Rule

In fact, if you try to use L'Hospital's Rule you will not get the right answer for this problem.

Lesson: only use L'Hospital's Rule when you have $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

We can extend this technique to cover other cases by <u>transforming back to $\frac{0}{0}$ or $\frac{\infty}{\infty}$</u>:

Case of $0 \cdot \infty$: use <u>reciprocals</u> to turn it into $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

<u>Example</u> $\lim_{x \to 0^+} x \ln x$

Check: $\left.\begin{array}{l} x \to 0 \\ \ln x \to -\infty \end{array}\right\}$ as $x \to 0^+$

so we have $0 \cdot (-\infty)$ ✓

Use reciprocals: $x \ln x = \dfrac{\ln x}{(1/x)}$

Now $\lim_{x \to 0^+} \frac{1}{x} = \infty$, so $\dfrac{\ln x}{(1/x)} \to \dfrac{\infty}{\infty}$, can use L'Hospital:

$$\lim_{x \to 0^+} \frac{\ln x}{(1/x)} = \lim_{x \to 0^+} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(1/x)} = \lim_{x \to 0^+} \frac{(1/x)}{(-1/x^2)}$$

$$= \lim_{x \to 0^+} (-x) = \boxed{0}$$

Case of $\infty - \infty$: Combine the terms:

Example $\lim_{x\to(\frac{\pi}{2})^-} \sec x - \tan x \quad (\infty - \infty)$

$$= \lim_{x\to(\frac{\pi}{2})^-} \frac{1}{\cos x} - \frac{\sin x}{\cos x} = \underbrace{\lim_{x\to(\frac{\pi}{2})^-} \frac{1-\sin x}{\cos x}}_{\text{That's } \frac{0}{0}!}$$

Use L'Hospital's Rule

$$\lim_{x\to(\frac{\pi}{2})^-} \frac{1-\sin x}{\cos x} = \lim_{x\to(\frac{\pi}{2})^-} \frac{\frac{d}{dx}(1-\sin x)}{\frac{d}{dx}(\cos x)}$$

$$= \lim_{x\to(\frac{\pi}{2})^-} \frac{-\cos x}{-\sin x} = \frac{-\cos(\frac{\pi}{2})}{-\sin\frac{\pi}{2}} = \frac{0}{-1} = 0.$$

Case of 0^0, ∞^0, 1^∞:

Take the <u>logarithm</u> and use $\lim_{x\to a} f(x) = e^{\lim_{x\to a}[\ln f(x)]}$

<u>Ex</u> $\lim_{x\to 0^+} x^x$

use $\ln(x^x) = x \ln x$

and $\lim_{x\to 0^+} x^x = e^{\lim_{x\to 0^+} \ln(x^x)} = e^{\lim_{x\to 0^+} x \ln x}$

we just need to find $\lim_{x\to 0^+} x \ln x$.

That's $0 \cdot (-\infty)$ and we did it already using L'Hospital's Rule

$\lim_{x\to 0^+} x \ln x = 0$

So $\lim_{x\to 0^+} x^x = e^0 = 1$