M408D

Discussion Session Aug 25, 2011

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Exercises:

Evaluate the following limits

8. $\lim_{x\to 1}\frac{x^a-1}{x^b-1}$

Solution: This limit is of the indeterminate form $\frac{0}{0}$, hence we apply l'Hospital's rule after which we can simply plug in

$$\lim_{x\to 1}\frac{x^a-1}{x^b-1} = \lim_{x\to 1}\frac{ax^{a-1}}{bx^{b-1}} = \frac{a}{b}$$

13. $\lim_{x\to 0}\frac{\tan(\mathrm{px})}{\tan(\mathrm{qx})}$

Solution: Similarly, we can apply l'Hospital's rule and then plug in

$$\lim_{x\to 0}\frac{\tan(\mathrm{px})}{\tan(\mathrm{qx})} = \lim_{x\to 0}\frac{p\sec^2(px)}{q\sec^2(qx)} = \frac{p}{q}$$

22. $\lim_{x\to 0}\frac{e^x-1-x-\frac{1}{2}x^2}{x^3}$

Solution: This limit is of the indeterminate form $\frac{0}{0}$ and in this case we may apply l'Hospital's rule 3 times until the denominator becomes a constant, at which point we may evaluate by plugging in

$$\lim_{x\to 0}\frac{e^x-1-x-\frac{1}{2}x^2}{x^3} = \lim_{x\to 0}\frac{e^x-1-x}{3x^2} = \lim_{x\to 0}\frac{e^x-1}{6x} = \lim_{x\to 0}\frac{e^x}{6} = \frac{1}{6}$$

25. $\lim_{t\to 0} \frac{5^t-3^t}{t}$

Solution: l'Hospital's rule is applicable once, then plugging in gives us our answer

$$\lim_{t\to 0} \frac{5^t-3^t}{t} = \lim_{t\to 0} \ln 5 \cdot 5^t - \ln 3 \cdot 3^t = \ln 5 - \ln 3 = \ln\frac{5}{3}$$

47. $\lim_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$

Solution: This limit is of the form$\infty - \infty$, so we can manipulate the function until it is of an indeterminate form to which l'Hospital's rule is applicable by finding a common denominator

$$\frac{x}{x-1} - \frac{1}{\ln x} = \frac{x\ln x}{(x-1)\ln x} - \frac{(x-1)}{(x-1)\ln x} = \frac{x\ln x - x + 1}{(x-1)\ln x} = \frac{x\ln x - x + 1}{x\ln x - \ln x}$$

So now we can apply l'Hospital's rule twice and then plugging gives us the limit

$$\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1}\left(\frac{x\ln x - x + 1}{x\ln x - \ln x}\right) = \lim_{x\to 1}\left(\frac{\ln x + 1 - 1}{\ln x + 1 - \frac{1}{x}}\right)$$

$$= \lim_{x\to 1}\left(\frac{\ln x}{\ln x + 1 - \frac{1}{x}}\right) = \lim_{x\to 1}\left(\frac{\frac{1}{x}}{\frac{1}{\mathrm{x}} + \frac{1}{x^2}}\right) = \frac{1}{1+1} = \frac{1}{2}$$

55. $\lim_{x\to 0}(1-2x)^{1/x}$

Solution: This limit is indeterminate of the form1^∞. Noticing that$y = e^{\ln y}$, we thus know that $\lim_{x\to 0}(1-2x)^{1/x} = e^{\lim_{x\to 0}\ln(1-2x)^{1/x}} = e^{\lim_{x\to 0}\frac{1}{x}\ln(1-2x)}$

And so we consider the limit on the exponent, which is of the form $\frac{0}{0}$ and hence we apply l'Hospital's rule once and then plug in

$$\lim_{x\to 0}\frac{\ln(1-2x)}{x} = \lim_{x\to 0}\frac{\left(\frac{-2}{1-2x}\right)}{1} = -2$$

Hence,

$$\lim_{x\to 0}(1-2x)^{1/x} = e^{\lim_{x\to 0}\ln(1-2x)^{1/x}} = e^{-2}$$

87. Let $\boldsymbol{S(x) = \int_0^x \sin\left(\frac{1}{2}\pi t^2\right) dt}$, and evaluate the following limit:

$$\boldsymbol{\lim_{x\to 0}\frac{S(x)}{x^3}}$$

Solution: As $x \to 0$, we have that $S(x) \to 0$, hence this limit is of indeterminate form $\frac{0}{0}$ and we may apply l'Hospital's rule 3 times, at which point we may plug in

Before doing so we notice that by the fundamental theorem of calculus:

$$(S(x))' = \sin\left(\frac{1}{2}\pi x^2\right)$$

And so

$$\lim_{x\to 0}\frac{S(x)}{x^3} = \lim_{x\to 0}\frac{\sin\left(\frac{1}{2}\pi x^2\right)}{3x^2} = \lim_{x\to 0}\frac{\cos\left(\frac{1}{2}\pi x^2\right)\pi x}{6x}$$
$$= \lim_{x\to 0}\frac{-\sin\left(\frac{1}{2}\pi x^2\right)\pi^2 x^2 + \cos\left(\frac{1}{2}\pi x^2\right)\pi}{6} = \frac{\pi}{6}$$