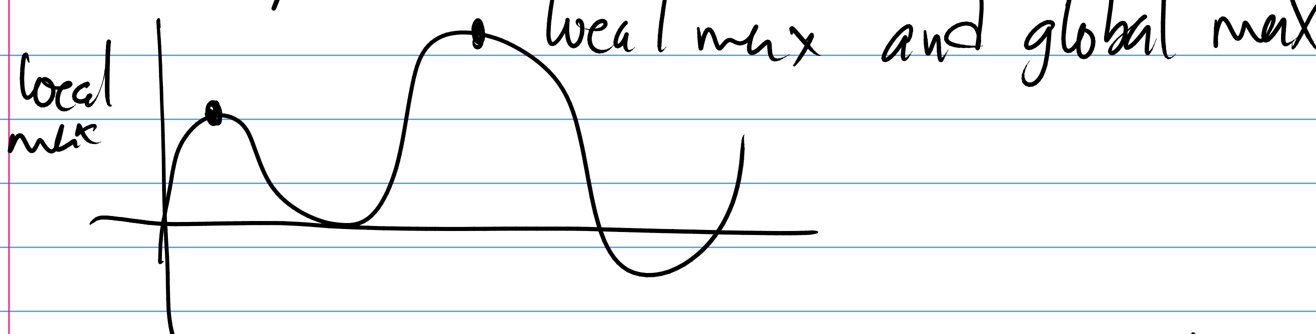
Global max/min

Last we saw that a local max/min always occurs at a critical point

$f(x,y,z)$

$$\text{Max/Min} \Rightarrow \nabla f = 0$$

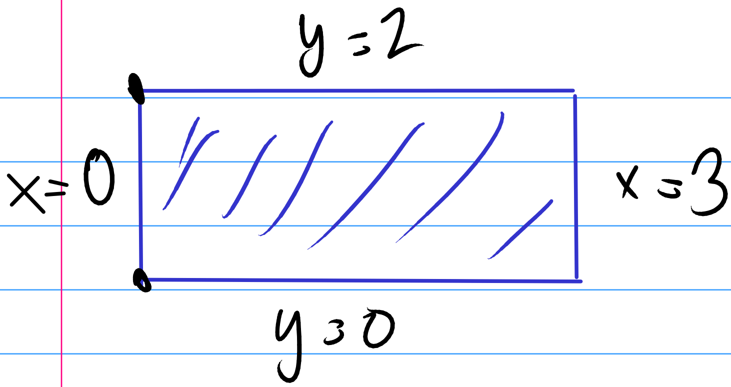


For Global max/min, you need to test all of the critical points and the boundary points

Ex $f(x,y) = x^2 - 2xy + 2y$

Maximize/Minimize on the Rectangle

$$D = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le 2\}$$



- Critical points in the interior
- "Boundary critical points"
- Corner points

Critical points in the interior.

$f(x,y) = x^2 - 2xy + 2y$

$\nabla f = \langle 2x - 2y, -2x+2 \rangle$

$\left.\begin{array}{l} 2x - 2y = 0 \\ -2x + 2 = 0 \end{array}\right\} \Rightarrow x = 1 \Rightarrow y = 1$

$(1,1)$ is critical point $f(1,1) = 1$

"Boundary max/min"

$\rightarrow$ left side $x=0$ $0 \leq y \leq 2$

$f(0,y) = 2y$ over the range $0 \leq y \leq 2$

$\left.\begin{array}{lll} (0,0) & \text{min} & f(0,0) = 0 \\ (0,2) & \text{max} & f(0,2) = 4 \end{array}\right\}$

Bottom: $y=0$, $0 \le x \le 3$

$f(x,0) = x^2$ $\quad$ $(0,0)$ min $f(0,0)=0$

$(3,0)$ max $f(3,0)=9$

Right $x=3$, $0 \le y \le 2$

$f(3,y) = 9-4y$ $\quad$ $(3,2)$ min $f(3,2)=1$

$(3,0)$ max $f(3,0)=9$

Top $y=2$, $0 \le x \le 3$

$f(x,2) = x^2-4x+4$

Look for critical points

$$\frac{d}{dx}(x^2-4x+4) = 2x-4 = 0$$

$$\Rightarrow x=2$$

which is in the range $0 \le x \le 3$

Need to consider $(2,2)$ as well

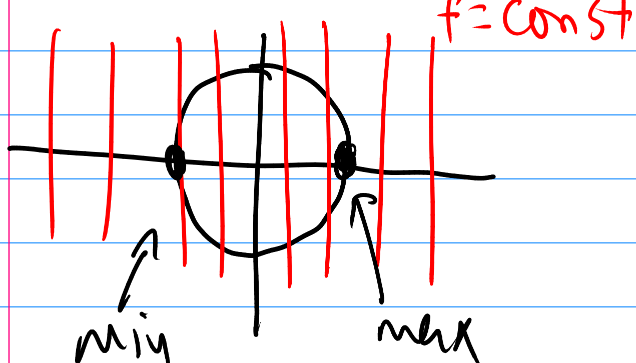
$f(2,2) = 0$

MAX $=9$ at $(3,0)$ $\quad$ MIN $=0$ at $(0,0)$ & $(2,2)$

Lagrange Multipliers

Problem: want to maximize/minimize a function $f(x,y)$ subject to a constraint $g(x,y)=k$

Eg Maximize $f(x,y)=x$ over the circle $g(x,y)=x^2+y^2=1$



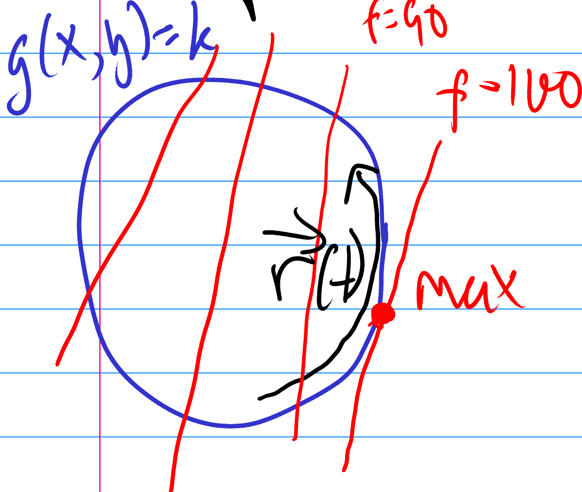
Theorem Assume $\nabla g \neq 0$ at any point of the constraint set $g(x,y,z)=k$

- At each max/min of $f(x,y,z)$ on the set $g(x,y,z)=k$, there is a number λ such that $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$

(Note you also have to solve for λ)

Proof in 2 dimensions

Suppose (x_0, y_0) is a max or a min of f subject to the constraint $g(x,y)=k$



Let $\vec{r}(t)$ be a parametric curve in $g(x,y)=k$

that is $g(\vec{r}(t)) = k$ for all t.

such that $\vec{r}(0) = (x_0, y_0)$

and $\vec{r}\,'(0) \neq 0$.

$$\frac{d}{dt}\left(g(\vec{r}(t)) = k\right)$$

$$\nabla g(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0 \quad \text{at } t=0 \quad \nabla g(x_0, y_0) \cdot \vec{r}\,'(0) = 0$$

On the other hand since $f(x_0, y_0)$ is an max on $g(x,y)=k$

the function $f(\vec{r}(t))$ has a local max

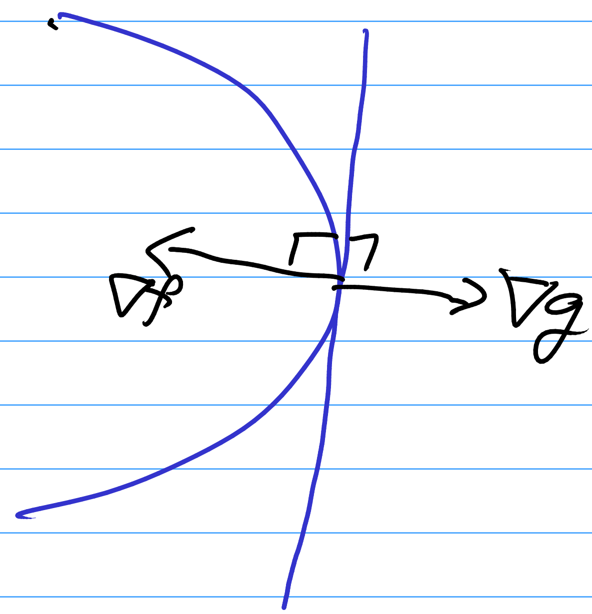
at $t=0$.

$$\left.\frac{d}{dt}\right|_{t=0} \left(f(\vec{r}(t))\right) = 0$$

$$\nabla f(x_0,y_0)\cdot \vec{r}\,'(0) = 0$$

Since $\nabla f(x_0,y_0)$ and $\nabla g(x_0,y_0)$ are perpendicular to the same vector $\vec{r}\,'(0)$, they must be proportional

$\Rightarrow \nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0)$ for some λ.



similar argument in 3D

Ex Max/min of

$f(x,y) = x^2 + 2y^2$ on circle $g(x,y) = x^2+y^2 = 1$

$\nabla f = \langle 2x, 4y \rangle$

$\nabla g = \langle 2x, 2y \rangle$

$\nabla f = \lambda \nabla g \quad \begin{cases} 2x = \lambda \cdot 2x \\ 4y = \lambda \cdot 2y \end{cases}$

unknowns : x, y, λ

$\nabla f = \lambda \nabla g$ gives 2 equations

$g(x,y) = k$ gives the third equation

$\begin{cases} 2x = \lambda \cdot 2x \\ 4y = \lambda \cdot 2y \\ x^2 + y^2 = 1 \end{cases} \Rightarrow x = \lambda x \Rightarrow \begin{cases} \lambda = 1 \\ \text{or} \\ x = 0 \end{cases}$

$\lambda = 1 \rightarrow 4y = 2y \Rightarrow y = 0 \Rightarrow x = \pm 1$

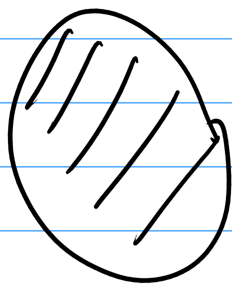
$x = 0 \Rightarrow y = \pm 1 \Rightarrow \lambda = 2$

(x,y)	$(0,+1)$	$(0,-1)$	$(+1,0)$	$(-1,0)$
$f(x,y)$	2	2	1	1

Max $= 2$ Min $= 1$

What about $f(x,y) = x^2+2y^2$

over the disk $\{x^2+y^2 \le 1\}$



- look for critical points in the interior $\nabla f = 0$
- use lagrange multiplers on the boundary

$$\nabla f = \lambda \nabla g, \quad g = k$$

Ex find point on the sphere $x^2+y^2+z^2=4$ closest to $(3,1,-1)$

Function to minimize = distance to $(3,1,-1)$

Constraint $x^2+y^2+z^2=4$