Iterated integrals

Exam topics

→ Lines and planes & Quadric Surfaces

→ Chapter 14 curves in 3-space

Tangents to curves velocity, acceleration, speed, arclength.

→ Chapter 15 partial derivatives

chain rule gradient tangent planes

max/min Lagrange multipliers

Iterated integrals

Last time $$\iint_R f(x,y)\,dA$$

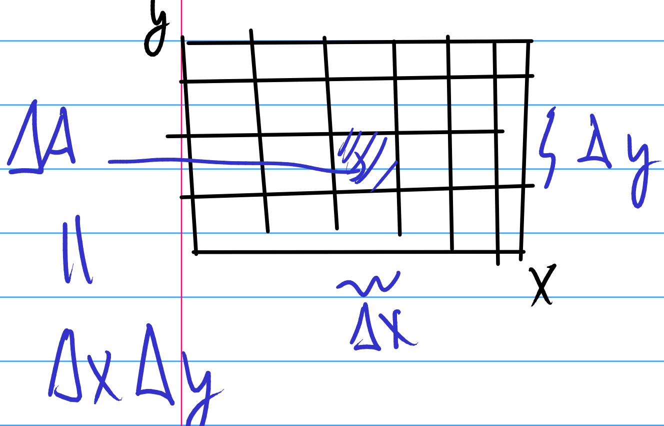
defined as a limit of Riemann sums
"volume under the graph"

Today: Compute exactly using Fundamental Theorem of calculus

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Can do this "twice" for double integrals. $f(x,y)$ function

$$dA = dx\,dy$$



$$R = [a,b] \times [c,d]$$
$$= \{a \le x \le b,\ c \le y \le d\}$$

$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dy\,dx$$

$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

dx dy

$$= \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

Iterated integral

First / Inside integral

$$\int_c^d f(x,y)\,dy$$

This is called partial integration with respect to y.

→ We hold x constant during the process of integration, and we regard the integrand as function of y only

→ When the integration is complete, we realize that x could have been any particular value, so the result is still a function of x

$$A(x) = \int_c^d f(x,y)\,dy$$

→ A(x) does not depend on y

We say that y has been "integrated out."

Second / Outer integral

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b A(x)\,dx$$

this is just an ordinary 1-var integral

Can also treat the integrals in the other order

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

we do the integrals from the inside out

Ex $\displaystyle\iint_R x\sin y\,dA$

$R = \underbrace{[0,2]}_{x} \times \underbrace{[0,\tfrac{\pi}{2}]}_{y}$

$$\int_0^2 \int_0^{\pi/2} x\sin y\,dy\,dx$$ Iterated integral

$$= \int_0^2 \left[\Big[-x \cos y \Big]_{y=0}^{y=\pi/2} \right] dx$$

$\uparrow$

inverse of $\frac{\partial}{\partial y}$

$$= \int_0^2 \left[-x \cdot 0 + x \cdot 1 \right] dx$$

$$= \int_0^2 x \, dx = \left[\frac{1}{2} x^2 \right]_{x=0}^{x=2} = \frac{1}{2}(2)^2 - \frac{1}{2} 0^2$$

$$= 2.$$

"Fubini's Theorem"

Double integral $\iint f \, dA$

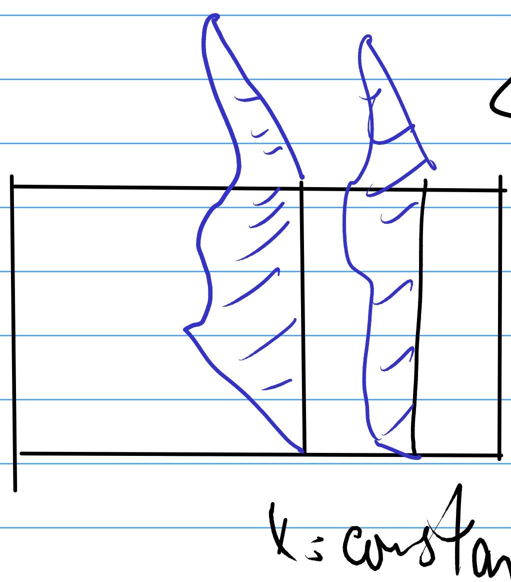
Iterated integral $\int_c^d \int_a^b f \, dx \, dy$

Iterated integral $\int_a^b \int_c^d f \, dy \, dx$

all equal.

<u>Idea</u> All are related to the volume under the graph.

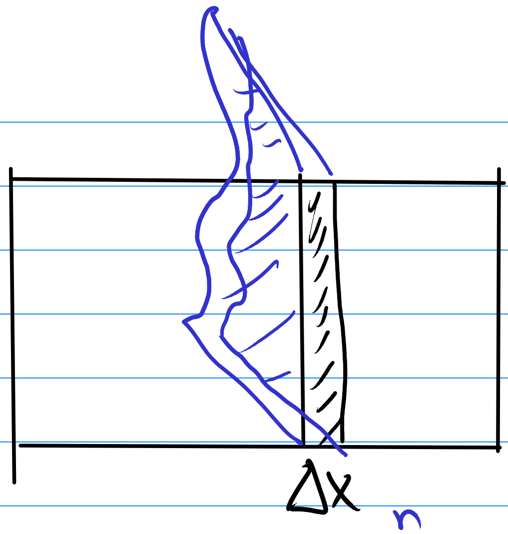
$$\int_a^b \int_c^d f(x,y) \, dy \, dx$$



area of a slice

$$= \int_c^d f(x,y) \, dy$$

$$= A(x)$$



volume of the thickened slice $\approx A(x)\Delta x$

$$\text{volume} = \sum_{i=1}^{n} A(x_i)\,\Delta x$$

$\Rightarrow$ in limit $\Delta x \to 0$

$$\text{Volume} = \int_a^b A(x)\,dx = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Ex $\int_1^2 \int_0^{\pi} y\sin(xy)\,dy\,dx$

$\int_0^{\pi} y\sin(xy)\,dy$

$u = y$

$dv = \sin(xy)$

$du = dy \qquad v = \dfrac{-\cos(xy)}{x}$

$$= -\frac{y\cos(xy)}{x}\Bigg|_{y=0}^{y=\pi} + \frac{1}{x}\int_0^{\pi} \cos(xy)\,dy$$

$$= \frac{-\pi\cos(\pi x)}{x} + \frac{1}{x}\left[\frac{\sin(xy)}{x}\right]_0^{\pi}$$

$$= \frac{-\pi\cos(\pi x)}{x} + \frac{1}{x^2}\sin(x\pi)$$

$$\int_1^2 \qquad dx$$

$$\int_0^{\pi} \int_1^{2} y \sin(xy)\, dx\, dy$$

$$\int_0^{\pi} \Big[-\cos(xy) \Big]_{x=1}^{x=2} dy$$

$$\int_0^{\pi} \big[-\cos(2y) + \cos(y) \big] dy$$

other order is much easier.