Integral Test

Over the course of the last two lectures, we have built an analogy between sequences and series on the one hand, and functions and integrals on the other.

function $f(x)$ (x a real number)	sequence a_n (n an integer)
limit $\lim_{x\to\infty} f(x)$	limit $\lim_{n\to\infty} a_n$
Improper integral $\int_1^\infty f(x)\,dx$	series $\sum_{n=1}^{\infty} a_n$
$\int_1^t f(x)\,dx$	sequence of partial sums $S_n = \sum_{i=1}^{n} a_i$
convergence: $\lim_{t\to\infty} \int_1^t f(x)\,dx$ exists	convergence $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$ exists

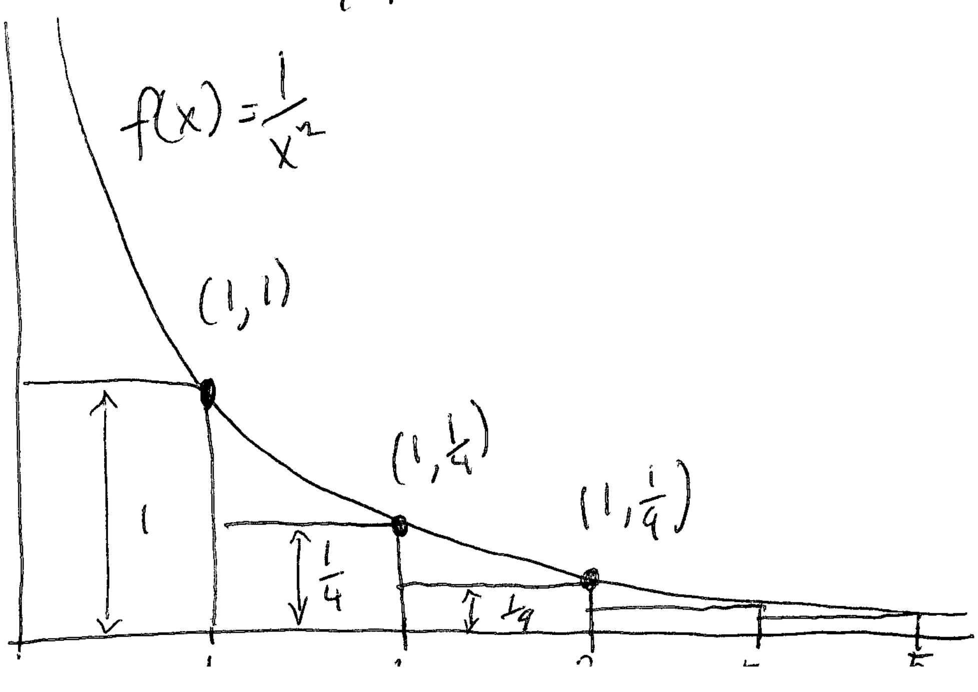
Today we will make a portion of this analogy more precise, by relating convergence of series to convergence of improper integrals.

We begin with two examples $\sum_{i=1}^{\infty} \frac{1}{i^2}$ and $\sum_{i=1}^{\infty} \frac{1}{i}$

<u>Example</u> Consider $\sum_{i=1}^{\infty} \frac{1}{i^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots$

Note that $\lim_{i \to \infty} \frac{1}{i^2} = 0$, so this <u>might</u> converge
(Cf. Test for divergence)

Idea: compare $\sum_{i=1}^{\infty} \frac{1}{i^2}$ and $\int_1^{\infty} \frac{1}{x^2}\,dx$



* The integral $\int_1^\infty \frac{1}{x^2}\,dx$ is the area under the curve to the right of 1.

* The series $\sum_{i=1}^{\infty} \frac{1}{i^2}$ is the area of the rectangles drawn under the curve.

* The picture proves that

$$\frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < \int_1^n \frac{1}{x^2}\,dx$$

(the first rectangle of height 1 lies to the left of $x=1$.)

Now we know that $\int_1^\infty \frac{1}{x^2}\,dx$ converges,

that is, $\lim_{n\to\infty} \int_1^n \frac{1}{x^2}\,dx$ exists, and $\frac{1}{4} + \frac{1}{9} + \cdots$

is less than this number!

therefore, $\frac{1}{4} + \frac{1}{9} + \cdots$ converges,

and so does $1 + \frac{1}{4} + \frac{1}{9} + \cdots$ converge.

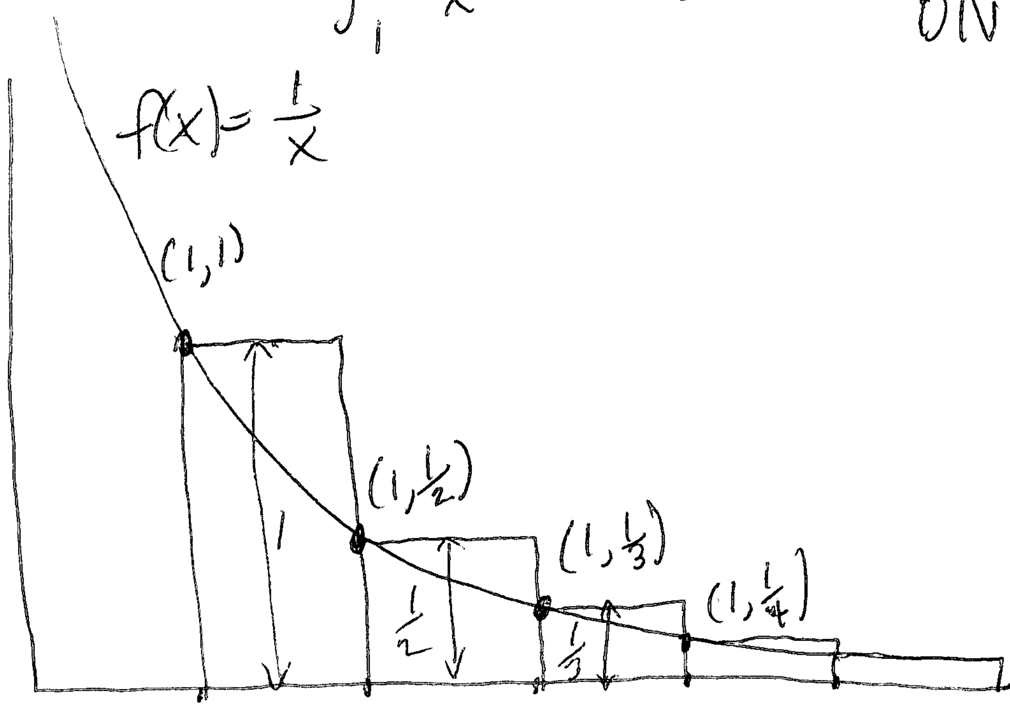
(in general, convergence of a series is equivalent to convergence of the "tail" of the series, that is, we can drop or add finitely many terms without affecting convergence.)

CONCLUSION: $\sum_{i=1}^{\infty} \frac{1}{i^2}$ converges

because $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges.

<u>Example</u> consider $\sum_{i=1}^{\infty} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$

compare with $\int_1^{\infty} \frac{1}{x}\,dx$ (which we know DIVERGES)



* the picture proves that

$$s_n = \sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{n} > \int_1^{n+1} \frac{1}{x}\,dx$$

since $\int_1^{\infty} \frac{1}{x}\,dx$ diverges, we know

$$\lim_{n\to\infty} \int_1^{n+1} \frac{1}{x}\,dx = \infty$$

since the partial sum $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is actually <u>bigger than</u> this divergent integral, the sequence s_n also diverges to ∞, and we say $\sum_{i=1}^{\infty} \frac{1}{i}$ DIVERGES.

<u>Theorem</u> (Integral Test)

* Suppose $f(x)$ is a <u>positive</u> <u>continuous</u> <u>decreasing</u> function, which is defined for $1 \le x < \infty$.

* Suppose the sequence a_i satisfies $a_i = f(i)$

THEN:

If $\int_1^\infty f(x)\,dx$ <u>converges</u> then $\sum_{i=1}^\infty a_i$ <u>converges</u>

If $\int_1^\infty f(x)\,dx$ <u>diverges</u> then $\sum_{i=1}^\infty a_i$ <u>diverges</u>.

Informally, "The series $\sum_{i=1}^\infty a_i$ and the integral $\int_1^\infty f(x)dx$ <u>converge</u> or <u>diverge</u> <u>together</u>."

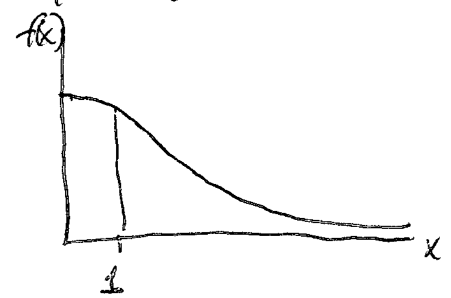
<u>Note</u>: the conditions that $f(x)$

is $\left\{\begin{array}{l}\text{POSITIVE}\\ \text{CONTINOUS}\\ \text{DECREASING}\end{array}\right\}$ are necessary in order for the integral test to work.

Ex Does $\sum_{i=1}^{\infty} \frac{1}{i^2+1}$ converge or diverge?

we have $a_i = f(i)$, where $f(x) = \frac{1}{x^2+1}$

$f(x)$ is positive, continuous,
and it is decreasing on $[1, \infty)$



Apply integral test

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \left(\tan^{-1}x\right)\Big|_1^t$$

$$= \lim_{t\to\infty} \left(\tan^{-1}t - \tan^{-1}1\right) = \lim_{t\to\infty} \left(\tan^{-1}t - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

The integral converges, so the series $\sum_{i=1}^{\infty} \frac{1}{i^2+1}$ does too.

<u>Ex</u> $\sum_{i=1}^{\infty} i e^{-i} \longrightarrow$ use $f(x) = xe^{-x}$

Need to look at $\int_1^{\infty} xe^{-x}\,dx$

But first, need to make sure $f(x)$ is positive, continuous, and decreasing, when $x \geq 1$

positive: ok because x is positive

continuous: yes

decreasing: $f'(x) = -xe^{-x} + e^{-x} = (1-x)e^{-x}$

$$(1-x)e^{-x} \leq 0 \quad \forall \quad x \geq 1,$$

So f is decreasing in this range.

Now $\int_1^t xe^{-x}\,dx = \left[-xe^{-x}\right]_1^t - \int_1^t (-e^{-x})\,dx$

$= \left[-xe^{-x}\right]_1^t - \left[e^{-x}\right]_1^t = -te^{-t} + e^{-1} - e^{-t} + e^{-1}$

$\lim_{t\to\infty} (-te^{-t} - e^{-t} + 2e^{-1}) = 2e^{-1}$ converges

So $\sum_{i=1}^{\infty} i e^{-i}$ converges as well.

Important example For which p does $\sum_{i=1}^{\infty} \frac{1}{i^p}$ converge?

In the lecture on improper integrals, we saw that

$$\int_1^{\infty} \frac{1}{x^p}\,dx \begin{cases} \text{converges if } p>1 \\ \text{diverges if } p\leq 1 \end{cases}$$

Consequently, the integral test combines with this statement to give:

$$\sum_{i=1}^{\infty} \frac{1}{i^p} \begin{cases} \text{converges if } p>1 \\ \text{diverges if } p\leq 1 \end{cases}$$

This series is called the p-series and it will turn out to be a useful fact that it converges if and only if $p>1$.