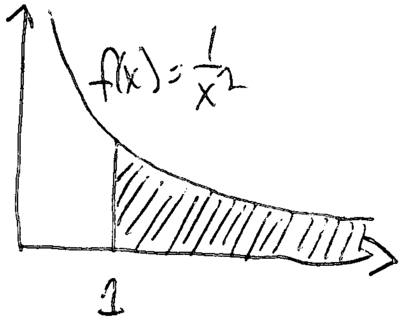
Improper Integrals

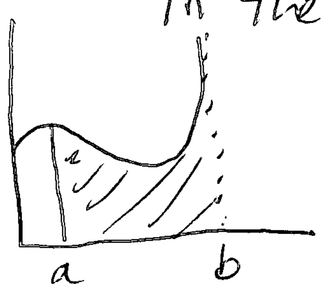
Type 1: Infinite limits of integration,
That is, integration over an unbounded interval.

For example $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$

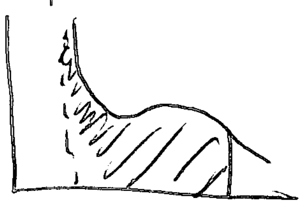


- $\displaystyle\int_a^{\infty} f(x)\,dx$, $\displaystyle\int_{-\infty}^{b} f(x)\,dx$, or $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$

Type 2: $f(x)$ has an infinite discontinuity in the ~~integral~~ interval of integration

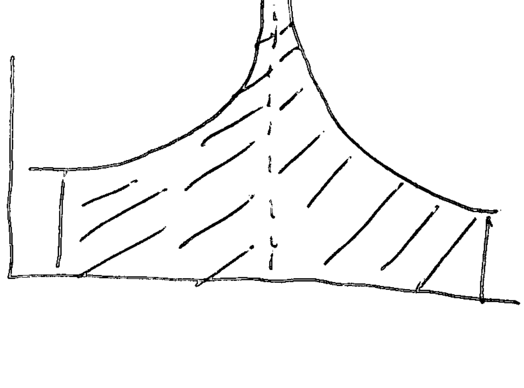


or



(dotted line indicates vertical asymptote)

Or

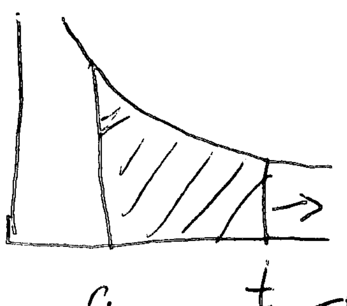


Improper integrals are difficult to define interms of Riemann sums, and, more importantly, it is quite possible for them to be infinite or to have no meaningful value.

Safest way to treat Improper integrals is as limits of proper integrals

(Proper integral = $\int$ of bounded function on a bounded interval)

<u>Type 1</u>: For $\int_a^\infty f(x)\,dx$, cut off at a finite t compute $\int_a^t f(x)\,dx$, and take limit as $t \to \infty$.



IF $\int_a^t f(x)\,dx$ exists, and

$\lim_{t\to\infty} \int_a^t f(x)\,dx$ EXISTS AND IS FINITE

THEN:

~~Then~~ we say $\int_a^\infty f(x)\,dx$ is CONVERGENT

and $\boxed{\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx}$.

If $\lim_{t\to\infty} \int_a^t f(x)\,dx$ is $+\infty, -\infty$, or does not exist for some other reason, then we say that $\int_a^\infty f(x)\,dx$ is DIVERGENT.

Same story for $\int_{-\infty}^b f(x)\,dx$

~~IF $\int_{-\infty}^b f(x)$~~ If $\int_t^b f(x)\,dx$ exists

and $\lim_{t\to-\infty} \int_t^b f(x)\,dx$ EXISTS AND IS FINITE

Then ~~T~~ THEN $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$

~~For~~ and $\int_{-\infty}^b f(x)\,dx$ is called CONVERGENT.

For $\int_{-\infty}^\infty f(x)\,dx$, we demand that both

$\int_{-\infty}^a f(x)\,dx$ and $\int_a^\infty f(x)\,dx$ are convergent, and

set $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx.$ [can use any a to split it up]

How about

$$\int_1^\infty \frac{1}{x^2}\,dx \, ?$$

$$\int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^t = -\frac{1}{t} + 1$$

$$\lim_{t\to\infty}\left(-\frac{1}{t}+1\right) = 1$$

EXISTS
AND IS FINITE

So: $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left(-\frac{1}{t}+1\right) = 1$

$\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ <u>Converges</u>!

How about

$$\int_1^\infty \frac{1}{x}\,dx \ ?$$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty}\Big[\ln x\Big]_{x=1}^{x=t}$$

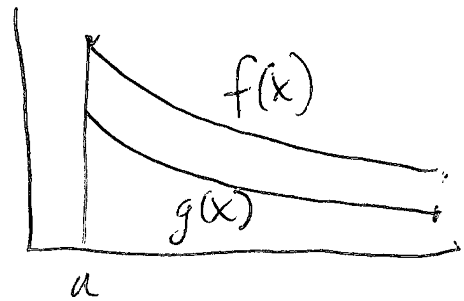
$$= \lim_{t\to\infty} \ln t = \infty.$$ So this integral DIVERGES.

In general:

$$\int_1^\infty \frac{1}{x^p}\,dx$$ converges if $p>1$ and diverges if $p\le 1$

Comparison Theorem:

Suppose $f(x) \geq g(x) \geq 0$



ⓐ If $\int_a^\infty f(x)\,dx$ converges

Then $\int_a^\infty g(x)\,dx$ converges

ⓑ If $\int_a^\infty g(x)\,dx$ diverges

Then $\int_a^\infty f(x)\,dx$ diverges

<u>Ex</u> $\int_1^\infty \frac{1+e^{-x}}{x}$ since $\frac{1+e^{-x}}{x} > \frac{1}{x}$

and $\int_1^\infty \frac{1}{x}\,dx$ diverges, we know that $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$

diverges as well.

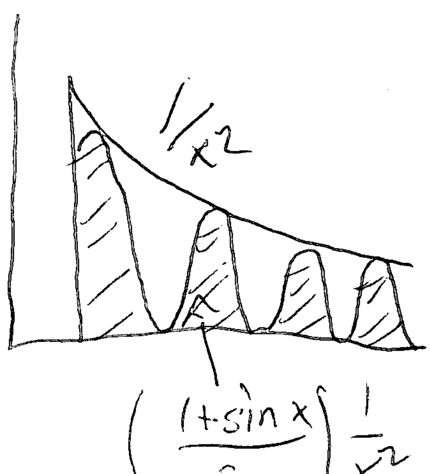
Comparison theorem can tell us something converges, even if we can't compute it.

<u>Example</u> $\displaystyle\int_1^\infty \left(\frac{1+\sin x}{2}\right)\frac{1}{x^2}\,dx$

$\sin x$ varies between -1 and 1,

so $\dfrac{1+\sin x}{2}$ is always between 0 and 1

Hence, $\left(\dfrac{1+\sin x}{2}\right)\dfrac{1}{x^2}$ is always between 0 and $\dfrac{1}{x^2}$



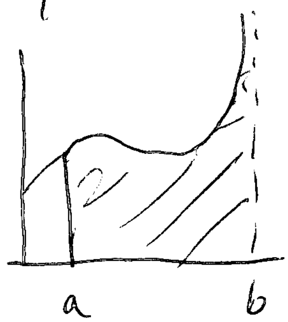
Comparison theorem tells us that since $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges,

$\displaystyle\int_1^\infty \left(\frac{1+\sin x}{2}\right)\frac{1}{x^2}\,dx$ <u>converges</u>.

Even though we don't know which finite number is the value of the integral, we know it exists.

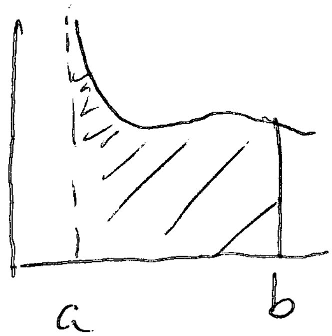
Type 2: infinite discontinuity

Use a one-sided limit at the problematic point



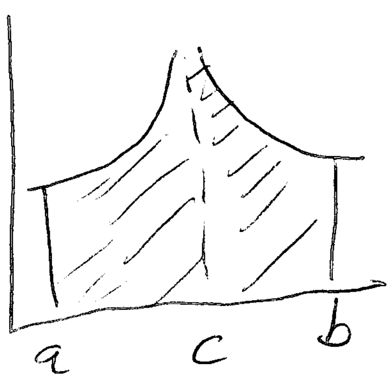
$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

provided the limit exists and is finite



$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

provided the limit exists and is finite.

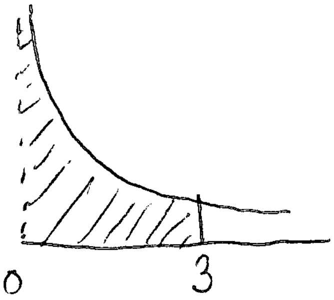


$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^a f(x)\,dx$$

$$= \lim_{t\to c^-} \int_a^t f(x)\,dx + \lim_{t\to c^+} \int_t^b f(x)\,dx$$

Break it up at the "bad point", and do ~~each~~ each part separately.

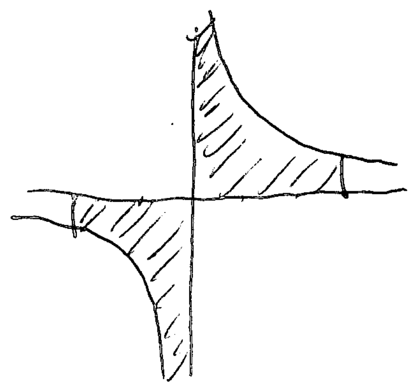
Ex $\displaystyle\int_0^3 \frac{1}{\sqrt{x}}\,dx$



$$= \lim_{t\to 0^+} \int_t^3 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} 2\sqrt{x}\Big|_t^3$$

$$= \lim_{t\to 0^+} \left(2\sqrt{3} - 2\sqrt{t}\right) = 2\sqrt{3}$$

Ex $\displaystyle\int_{-2}^3 \frac{1}{x}\,dx$



Break it up at $x=0$

$$\int_{-2}^0 \frac{1}{x}\,dx + \int_0^3 \frac{1}{x}\,dx = \lim_{t\to 0^-} \int_{-2}^t \frac{1}{x}\,dx + \lim_{t\to 0^+} \int_t^3 \frac{1}{x}\,dx$$

Do: $\displaystyle\lim_{t\to 0^-} \int_{-2}^t \frac{1}{x}\,dx = \lim_{t\to 0^-} \ln|t| - \ln|-2| = -\infty$

DIVERGENT $\Rightarrow$ We're done we know $\displaystyle\int_{-2}^3 \frac{1}{x}\,dx$ diverges

What about: $\displaystyle\int_{-2}^3 \frac{1}{x}\,dx = \ln|x|\Big|_{-2}^3 = \ln 3 - \ln 2$

WRONG! FTC only applies to proper integrals!